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Poly-Universe in School Education

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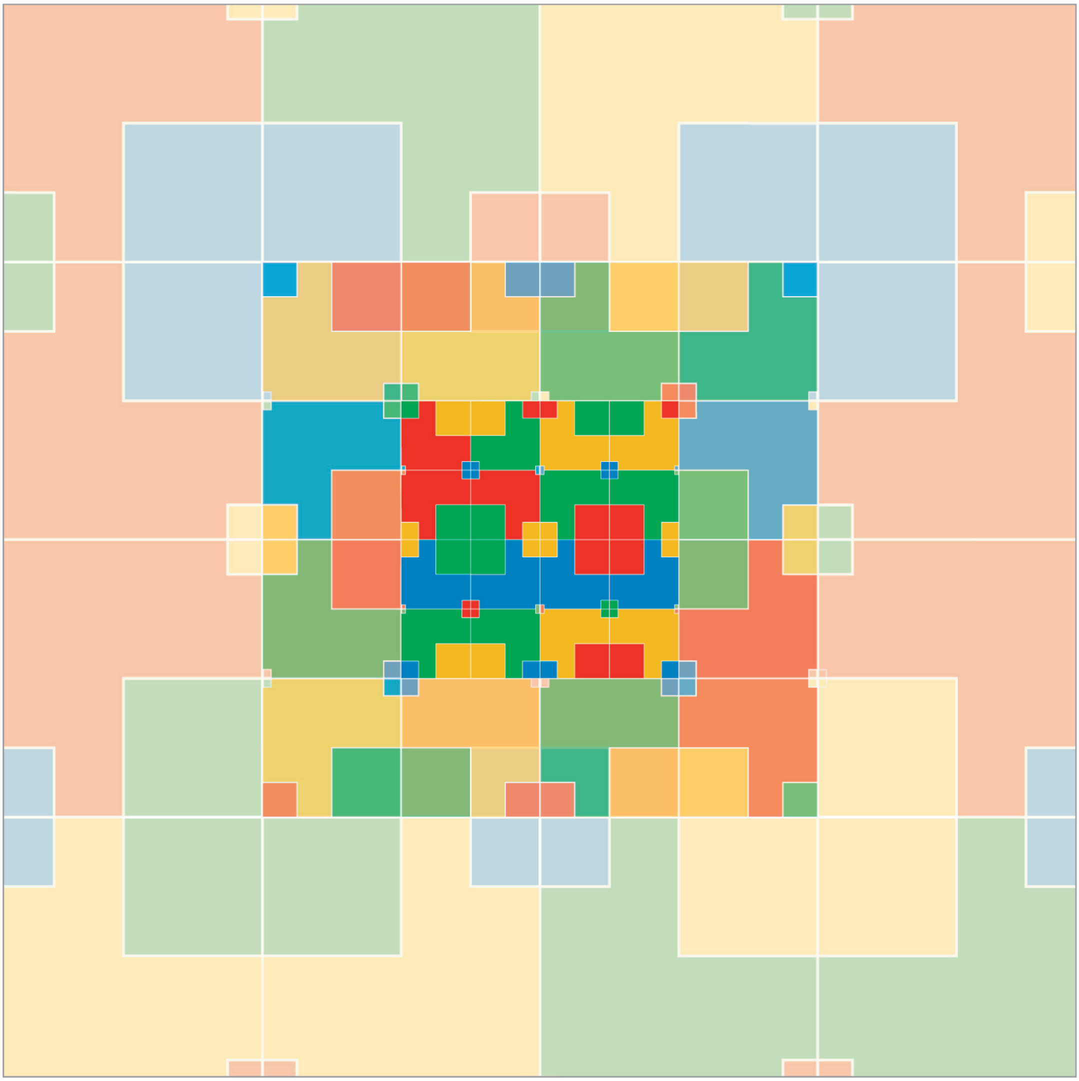

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POLY-UNIVERSE IN SCHOOL EDUCATION

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János Szász SAXON

assisted by
Szabina Tóth

A thematic issue

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Poly-Universe in School Education

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EDITORIAL

PUSE METHODOLOGY POLY-UNIVERSE IN SCHOOL EDUCATION

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Abstract: *The main objective of our Erasmus+PUSE project was to develop a new visual educational system for mathematics: the PUSE (Poly-Universe in School Education) Methodology, Visual Experience Based Mathematics Education 2019. The PUSE was based on the Poly-Universe game, which is a geometric skill-developing game by János Szász SAXON Hungarian fine artist. The novelty value of Poly-Universe lies in the scale-shifting symmetry inherent in its geometric forms and a colour combination system, which can be used universally and impact the educational system, particularly in the education of mathematics: geometry, sets and logic, combinatorics and probability, graphs and algorithms etc. The complexity emerging from its simplicity makes it more than a game, more than art, more than mathematics: these elements come all together – creating synergy in education……*

Keywords: Saxon, Poly-Universe, PUSE, Game, Mathematics Methodology, Visual Experience, Combinatorics, Scale shifting symmetry[1]

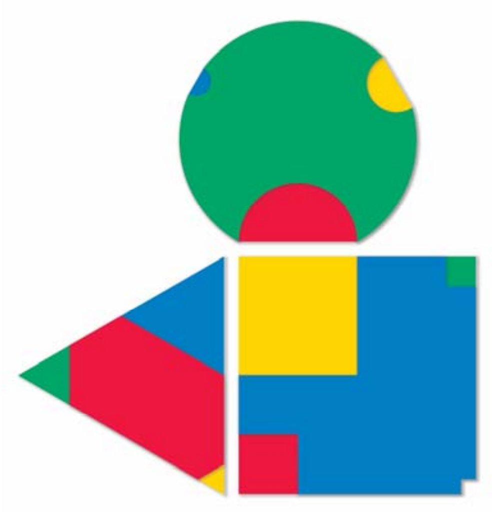

Fig: Saxon Poly-Universe Game, basic elements

1 THE POLY-UNIVERSE GAME IN MATHEMATICS EDUCATION

Today's children have a new kind of relationship with the world. They are citizens of the "global village", living in the endless web of connections, interconnections, and intersections. As a result, the educational system as a whole, including the teaching profession and teacher education, is challenged. New questions require new answers. The system of expectations and compliances, where teachers and children used to be caught up in, is getting looser and the student-teacher relationship in many cases turns into fruitful cooperation. Teachers aiming at successful collaborations, search for new tools and methods everywhere in the world. The present publication, the PUSE

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[1] The scale-shifting symmetry here refers to the smaller or larger size or proportion of things; it defines the scale of forms. Consequently, by 'Poly-Universe' we mean a system of forms in which forms appear in various sizes and on various scales, without, however, losing their original features. This is possible because in this system, forms are independent of scale changes (they are scale-invariant). In this way, the forms of a poly-dimensional world resemble those scale-invariant mathematical objects which have recently been called "fractals." The major difference lies in the fact that the discovery of these forms is in this case due to artistic devices, following the traditions of constructivism in visual art by János Saxon Szász. (Géza Perneczky, Art Historian, Dimension Crayon, 2000)

educational methodology, which is based on the Poly-Universe skill development tool, presents one possible solution in this process.

The Poly-Universe geometric skill development tool was invented by visual artist János Szász SAXON. The novelty of the game lies in the scale shift symmetry[1] of the basic shapes (circle, triangle, square) and in the colour combination system attached to it. Although the tool consists of truly simple elements and attributes (basic shapes, basic colours, proportions), it is at the same time extremely complex. Due to shifting proportions and changing colour combinations and their relationships, the number of possible solutions becomes virtually endless. That is why the Poly-Universe is more than a game, more than art, more than maths, it includes all: synergy in education…

The Erasmus+ PUSE (Poly-Universe in School Education) project[2] (Darvas, 2019) was the first step to explore how Poly-Universe's artistic-scientific approach could enter the world of mathematics education. Its necessity was proven by workshops held in dozens of countries, by the participation in math-art conferences[3] and last but not least by a research project of the Hungarian Academy of Sciences[4].

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[2] Erasmus+ PUSE (Poly-Universe in School Education) 2017−1−HU01−KA201−035938 Project.

[3] Bridges Conferences (http://bridgesmathart.org); ESMA−European Society for Mathematics and Art (http://www.math-art.eu); Experience Workshop events (www.experienceworkshop.org); RISD: 3rd Biennial Design Science Symposium on "Nature, Geometry, and the Symmetry of Space: Tetrahedron Discovers Itself and Universe" 2011 at RISD (Rhode Island School of Design) USA; Salon des Réalites Nouvelles, Paris

Mathematics is one of the key competencies. The teaching of primary and secondary school mathematics as an abstract science, which involves traditional tools and methods – is outdated. Many have recognized how important it is to raise children's curiosity and motivation, to be interactive, and make use of experience-centred education. Imagination and discovery claim more space, while students need more encouragement in mathematical problem solving so that they can recognize connections between visual and tangible patterns -instead of submerging in the abstract world of numbers – and find creative, playful solutions. Teaching to see, learning through sight!

2 PUSE METHODOLOGY BOOK AND WORKSHEETS FOR STUDENTS

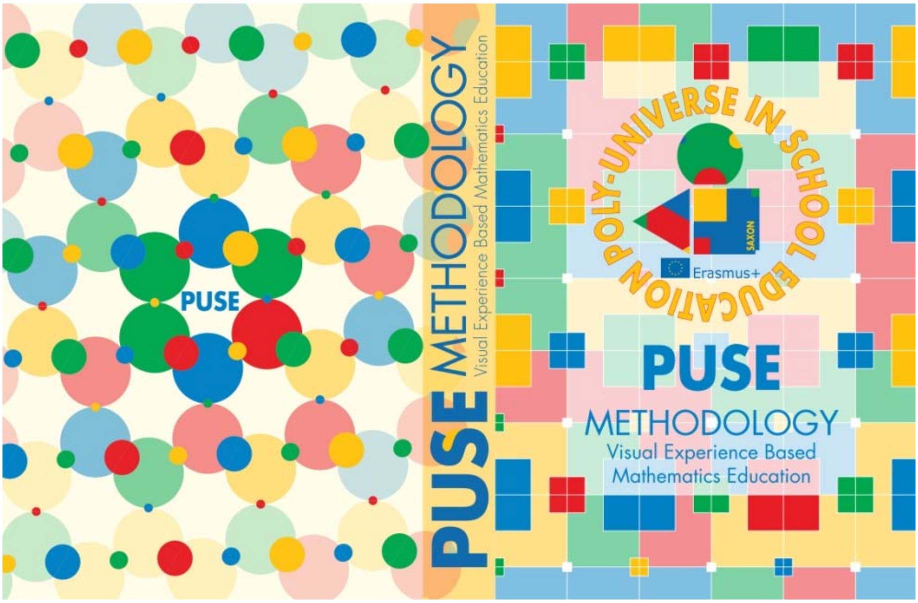
Erasmus+ PUSE's goal was to develop a geometric-combinatorial educational method, primarily for mathematics education, based on a visual system. The outcome of the project is an educational methodology and student's workbook, published both in Hungarian and in English, moreover an expandable online source of exercises. (www.poly-universe.com)

2.1 The PUSE Methodology book (Szász SAXON, Stettner, 2019) consists of five main thematic units:

1 Geometry & Measurement
2 Combinatorics & Probability
3 Sets & Logic
4 Graphs & Algorithms
5 Complex & Visuality

(https://www.realitesnouvelles.org); Saxon Art Gallery, Budapest (www.saxonart-gallery.com); Symmetry Festivals (http://festival.symmetry.hu); Synergetics Collaborative (SNEC), 2011 Boston Cambridge, USA (http://SynergeticsCollaborative.org); ULB−Université Libre de Bruxelles (https://ecolebelge.org/art-et-math)...

[4] Complex Mathematics Teaching in the 21st century – Developing combinatorics thinking based on recent findings. (Hungarian Academy of Sciences/MTA 2015 – Project manager Ödön Vancsó, head of ELTE Mathematics Teaching and Methodology Center)



2.2 Poly-Universe workshops and conferences preceding the Erasmus+ research made it clear that Poly-Universe does raise the interest in a wide spectrum – from kindergarten to university one might say. Based on this experience, the PUSE educational methodology and student workbook target three different age groups: primary school lower grades, upper grades, and high school students.

2.3 Our goal was to compile exercises and exercise series which encourage further thinking and raise new questions – instead of closing all thoughts after the solution is found. The Poly-Universe, rooted in a special artistic vision, does not only inspire mathematics but also triggers interdisciplinarity. Without affecting either Poly-Universe's artistic background or its physical characteristics, the use of the free online mathematical application called GeoGebra in the book expands the use of Poly-Universe in mathematics. (www.geogebra.org)

3 PUSE TEACHERS, ARTISTS, AND PSYCHOLOGISTS

The PUSE methodology book was created by a group consisting of teachers, artists, a psychologist, a sociologist, the inventor, and external experts. The four partner organizations (Finnish, Hungarian, Slovak, Spanish) come from different regions of Europe and operate in different educational systems. All contributing teachers have a good sense of the messages of our age, are forward-looking and support passages between arts and math. Thanks to the international partnerships, we learned how students and teachers approach problem-solving based on Poly-Universe's visual system in a Scandinavian-type educational system; in a Western-European environment based on the PISA method; in a strongly mathematics-centred Central-European educational system; and in the specific circumstances of minority schools. All participating schools gave valuable contributions to the development of the methodology and the exercises.

3.1 Erasmus+ PUSE (Poly-Universe in School Education) project Partners

Coordinator: Poly-Universe Ltd, Szokolya, Hungary; www.poly-universe.com

NetCoGame GamefulLiving Research Center Nonprofit Ltd., Budapest www.netcogame.com

Budapest Fazekas Mihály Practicing Primary School and Grammar School, Hungary; www.fazekas.hu

Nafarroako Ikastolen Elkartea Pamplona, Comunidad Foral de Navarra, Spain www.nafarroakoikastolak.net

Základná skola Gergelya Czuczora, Nové Zámky, Slovakia; www.czuczora.eu

Experience Workshop, Jyväskylä, Keski-Suomi, Finland www.experienceworkshop.org

Kaposvár University, Hungary; www.english.ke.hu

3.2 Cooperation between Students

Going beyond educational aspects, the need arose to map connections between the tool and psychology, so we included an expert in game research in the project as well.

The results of the measurement show that Poly-Universe effectively develops cognitive areas, such as visual perception and related skills. There is also a traceable increase in attention capacity. Moreover, based on surveys and interviews with experts, the impact of the tool manifested in areas such as creativity, mathematical and logical thinking or social skills. Using the tool in group work helps to map ingroup processes as well. In the course of observations – intentionally or not – participants' roles and the quality of their cooperation became manifest.

4 THE FUTURE OF PUSE AS AN ONLINE PLATFORM

The final output of our educational methodology is an interactive PUSE online system, where one can use the advantages of ICT based teaching.

The PUSE methodological book doesn't claim completeness, i.e. it is not a closed collection of exercises. This couldn't have been its purpose. The Poly-Universe, in fact, is not simply a tangible tool or a closed-system puzzle, but an open math-art system which models the Universe, allowing a glimpse into its internal laws. This book is a beginning, just like Christopher Columbus's undertaking once was, who was the first to cross the ocean, to let others following his footsteps discover America.

The PUSE methodology book opens up the way for teachers and students, artists and scientists. We can all enter Poly-Universe's spacious world, explore its complex realm, further discover its laws and regulations. While Poly-Universe offers scale shift, the methodology based on the tool offers a change of attitude for today's educational system – not in order to achieve small, medium or large bits of knowledge, but endless knowledge...

4.1 The main goal was to develop a PUSE interactive surface for teachers,

who can learn how to use the PUSE methodology in their classes. In order to reach the broader public, we published the analogue PUSE Teacher's book and Exercise tasks in English and in Hungarian on the internet. In the future, when the PUSE methodology book will be translated into several languages in EU education systems, other language versions will be available.

4.1.1 PUSE Methodology

Cf., (Szász SAXON, Stettner, 2019)

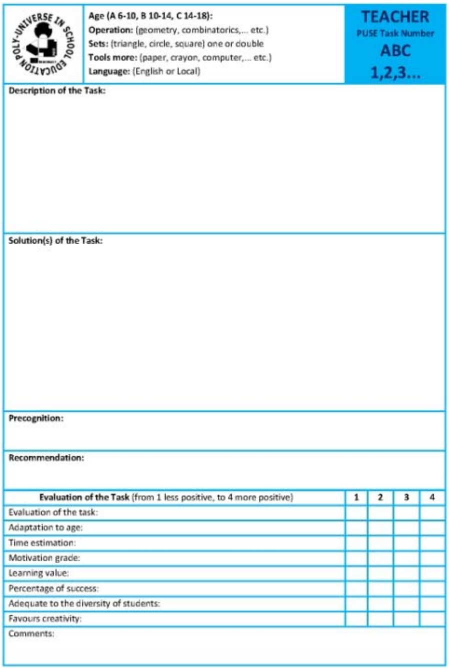
4.1.2 PUSE Exercise tasks in English, separate downloadable PDF files are available:

101-143 Geometry & Measurement
201-236 Combinatorics & Probability
301-319 Sets & Logic
401-416 Graphs & Algorithms
501-524 Complex & Visuality

4.2 PUSE Study published downloadable on 175 pages in English.

4.3 Downloadable PUSE Templates for Teachers/Students.

We created the possibility for other experts, teachers and students to make their own exercises and share them with the Poly-Universe community, thus expanding the PUSE methodology. Downloadable PUSE Templates for Teachers/Students are available in English and Hungarian versions (4 files). Another possibility is to teachers to give homework to students via the online system; the students could send the homework results to the teacher for checking. The downloadable interactive tasks are available for the students for collaborative or competitive use.



Age (A 6-10, B 10-14, C 14-18):
Operation: (geometry, combinatorics,... etc.)
Sets: (triangle, circle, square) one or double
Tools more: (paper, crayon, computer,... etc.)
Language: (English or Local)

TEACHER
PUSE Task Number
ABC
1,2,3...

Description of the Task:

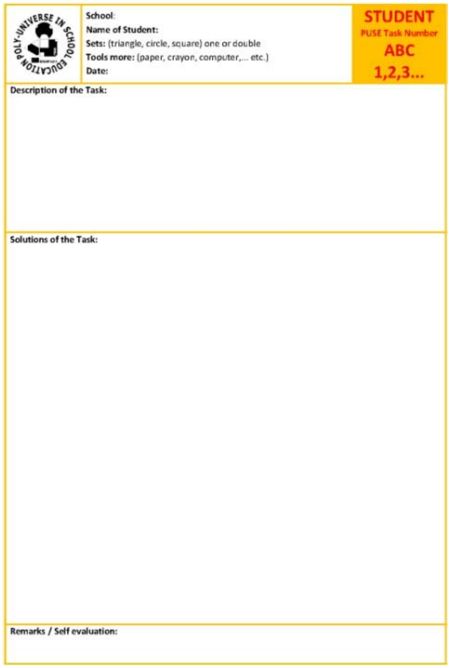
Solution(s) of the Task:

Precognition:

Recommendation:

Evaluation of the Task (from 1 less positive, to 4 more positive)	1	2	3	4
Evaluation of the task:				
Adaptation to age:				
Time estimation:				
Motivation grade:				
Learning value:				
Percentage of success:				
Adequate to the diversity of students:				
Favours creativity:				

Comments:



School:
Name of Student:
Sets: (triangle, circle, square) one or double
Tools more: (paper, crayon, computer,... etc.)
Date:

STUDENT
PUSE Task Number
ABC
1,2,3...

Description of the Task:

Solutions of the Task:

Remarks / Self evaluation:

4.4 PUSE animations and interactive applications.

The partnership created 8 animations and 4 interactive applications about the Poly-Universe Game, connected with other possible digital tools and platforms such as GeoGebra, Google Draw, Scratch, Ipad etc. It is impossible to ignore the fact that children nowadays live part of their lives in the digital space. With the help of GeoGebra, we have developed interactive programs for the design and further considering (such as a three-dimensional extension) of the spatial-geometric elements reflecting on the artistic backgrounds of Poly-Universe. But we can as well use Google Draw for free play, or for the digital modelling of exercise solutions which involve more Poly-Universe sets, or even for creating projects inspired by fractal geometry – to broaden children's perspective.

4.5 The PUSE-Forum

got established for everybody who wants to connect to other PUSE users actively, show something for others, share ideas or create an event about the methodology, or want to create some other new tasks for methodology etc. There are several hundred registered users before the end of the project.

4.6 The PUSE events' documentation

and photos and some older documentation about the Poly-Universe history got archived. The PUSE methodology contents for the online platform got developed by the same group of people who created the analogue, paper-based Teacher's book and Exercise tasks. The PUSE online system and the downloadable TREASURY is available free of charge for the public, for schools and professional institutions after registration: http://www.poly-universe.com.

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SYMMETRY IN EDUCATION

DISCOVERING INFORMATION VISUALIZATION THROUGH POLY-UNIVERSE

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Abstract: *Information visualization is the science and art of visualizing information that flows around us in many disciplines, intending to make the data more digestible and understandable for non-expert users. The objective of this paper is to introduce a new and somewhat surprising application of the Poly-Universe system, as an educational tool of understanding basic information visualization principles, and filling the symmetric and asymmetric constellations with content-dependent meaning through information-related story-telling.*

Keywords: information visualization, Poly-Universe, PUSE, story-telling.
MSC 2010: 00A66, 97U60

1 INTRODUCTION

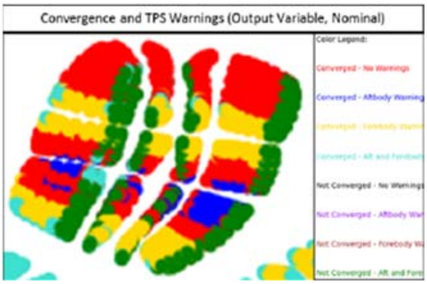
In today's society, data and their understanding and interpretation have become of central importance in many fields of science as well as in everyday life. Consequently, the correct and fast interpretation of raw information data is a new essential skill that needs to be developed already from early school years. The notion of visualization literacy, as a concept of the ability to confidentially create and interpret visual representations of data, is recently introduced and studied (Alper et al., 2017). The lack of visualization literacy can yield limited access to data and information, that can finally prevent youths and adults from informed decision making. Images are so important in nowadays life, that the need for fundamental changes in education is obvious in this respect (Nyíri, 2013). Understanding images and understanding data represented/visualized by these images and forms are especially important in the light of the recent European initiative of encouraging open data and open science. However, currently, we have limited educational resources and tools to effectively develop these skills (Saddiqa 2019).

In this paper, we initiate a new method of teaching and studying of information visualization with the help of Poly-Universe, a general-purpose artistic and mathematical system of forms developed by Saxon Szász (Saxon Szász 2010). This system is known to effectively support the development several skills, such as form and colour senses; abstractive vision; art sensitivity; complex logical thinking; or mathematical and combination skills. It can widely be used in school education, as demonstrated through several experience workshops (Saxon Szász - Dárdai 2019). Here we provide a new chapter of educational application of this system, by applying Poly-Universe to develop information visualization skills, the ability of understanding data and their interrelations, specifically the potential semantic aspects of symmetry and asymmetry. The method – with proper adjustment of problems similar to the ones discussed in this paper – is suitable for any level of education, from early childhood to university.

2 POLY-UNIVERSE SHAPES AND INFORMATION VISUALIZATION

Professional Information Visualization techniques are supported by various tools and software, but there are some essentials, that one can find in every tool: creating basic shapes representing basic data, using different colours for different data/shapes, and visualizing interrelations of data by some connections of these basic shapes. It is of utmost importance to express the symmetry or antisymmetry of these interrelations (e.g. balanced or unbalanced trade of goods between two countries) by some of the above-mentioned tools. We will be exploring this aspect in Section 3. We will prove through examples that Poly-Universe elements possess all the necessary properties and variety to be fit for this task.

Observing some real-life information visualization examples randomly selected from the scientific literature (see Figure 1), one can easily associate them with the shapes, colours and interrelations of the basic shapes of Poly-Universe. This fact inspired us to study the potential use of Poly-Universe in learning and teaching the fundamentals of information visualization.



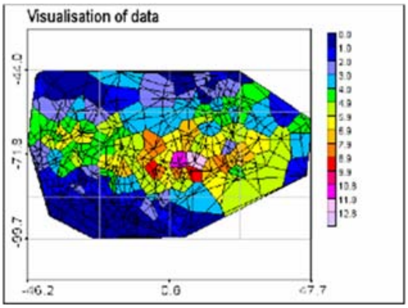


Figure 1: Inspirational examples of real-life information visualization figures about machine learning (left, Liles 2014), about social media management (middle, a software-generated example by Visme (visme.co)), and geospatial data (right, Kanevski 2008).

3 EDUCATIONAL SCENARIOS

Two different educational scenarios will be discussed in this section. In the first case, a construction created by Poly-Universe shapes is given, and students have to interpret and analyze this form using a pre-defined or jointly agreed meaning of colours, shapes and relations. This scenario is called interpretative.

In the other, somewhat reverse case, only the „story" (i.e. a set of data) is given, and students have to create its visualization using the Poly-Universe shapes, meanwhile explaining what colours, shapes and relations mean in their visualization. This scenario is called creative.

In the next subsections, we provide some examples for both scenarios, emphasizing the interpretation/creation of symmetric and asymmetric forms.

3.1 The interpretative scenario

Problem 1. Consider the following squares of Figure 2 (Saxon Szász – Dárdai, 2019) as a visual representation of countries. Colours of the square illustrate the four main products of the country: wheat (yellow), wood products (green), electrotechnical products (blue), clothing (red). In the top row, products of separate countries are visualized. In the bottom figures, neighbouring countries are presented. Two countries are in export-import relation if two squares share a common side or vertex. The type of goods to change is represented by colours along this side or having that vertex.

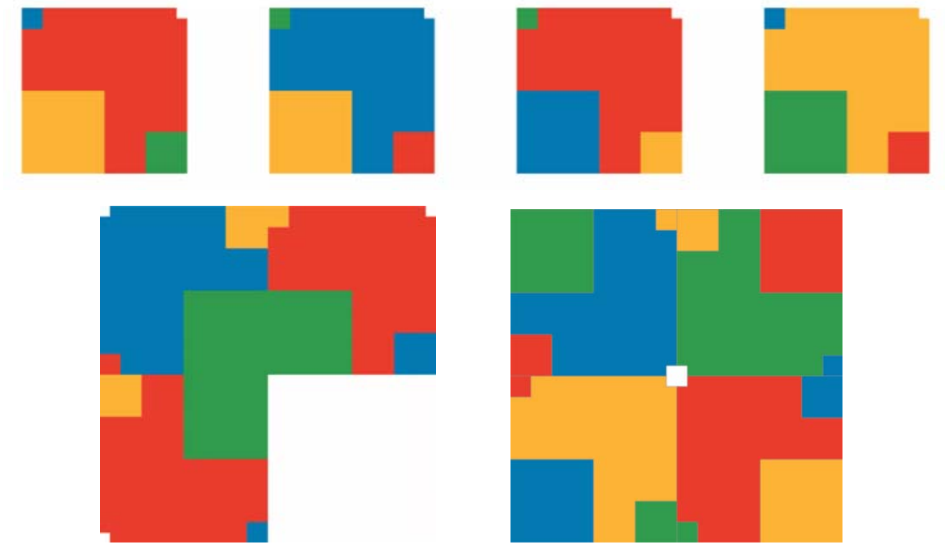


Figure 2: A country is represented by a square, together with the main products of the country (colours). Neighbouring countries are in export-import relation.

Potential questions to interpret the countries and their products:

- Which country in the top row produces the highest amount of electrotechnical products? Which country produces more wood products, the first or the second?

- In the bottom left figure, do the countries have symmetric export-import relation in terms of woods? What is the situation in terms of wheat? Find countries producing the same amount of wheat! Do they export wheat to each other? Make statements of the form „country A and country B produce the same amount of good X".

- In the bottom right figure, the two countries in the right-side change electrotechnical products in an asymmetric way. Find similar relationships with analogous amounts of change of other products between other countries!

- Let us call the economy of a group of countries „healthy" if they have an overall balanced production in some of these four products. Which group of the bottom row is healthier? Is a configuration with rotational symmetry always perfectly healthy? Can we complete the left group with a country from a perfectly healthy group of 4 countries?

Problem 2. Consider the following tessellations in Figure 3 (Saxon Szász – Dárdai, 2019) built from Poly-Universe triangles. Districts of the cities are coloured according to various political preferences. Different colours mean votes for different political movements (Yellow Party, Blue Party etc.).

Figure 3: Territories of two abstract cities (built from Poly-Universe triangles) are coloured according to political preferences: green, yellow, red and blue are for the 4 main political movements in the country

Potential questions to interpret the cities, districts and their political preferences:

- Which political movements are the winners of the voting procedure in the two cities?

- Do they have to initiate a coalition, or is there a single winner? Can two parties form a coalition with the majority? Can three parties form a coalition with the majority?

- Does axial symmetry of yellow, blue and red districts in left yields equal votes for the three parties?

- Surveys show that physical workers usually vote for the Red Party. Where can be the main district of factories in the city on the right side?

- Surveys show that people with high income usually live in the very centre of the city. Which is the favourite party of wealthy citizens in these cities?

3.2 The creative scenario

Problem 3. Consider the following story. There is a social event, a dinner at a scientific conference. People are sitting around three round tables, and – as it is usual in such events – they are talking to their neighbours. Of course, the extent of these conversations can vary from neighbours to neighbours. At the left table, the couples listen and talk to each other to the same extent pairwise, but they do not find a common theme. At the right table, each pair of neighbours finds an interesting joint field to talk about, but people sitting on the left side of the pair talk much more to his/her right couple than vice versa. Finally, in the middle table, pairs find a good common topic to discuss, and they mutually respect and listen to the view of each other, leading to a fruitful conversation. Task: represent the situation by some information visualization tool!

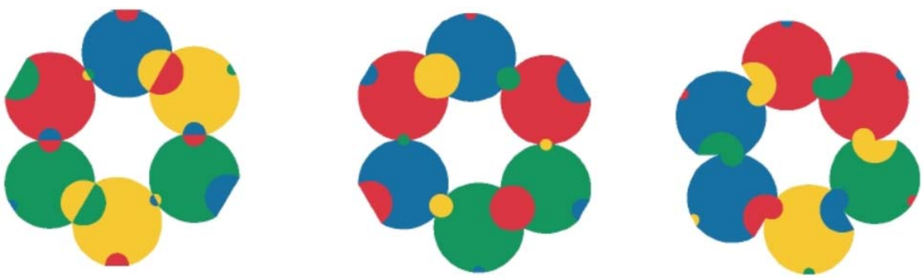


Figure 4: Pairwise conversation between pairs of people sitting around three tables (built from Poly-Universe circular elements). Colours denote different fields, the size of joining parts represent the extent of conversation and its symmetry, mutuality (in case of circular joining parts at the first two tables) or its asymmetry (in case of table 3)

One possible solution can be seen in Figure 4 using the circular elements of Poly-Universe (Figures from Kis, 2017 – note that we intentionally use existing figures to demonstrate that story-telling works for any kind of configuration). People are represented by circles, conversations and their extent are represented by the joining parts. If these joining parts are of the same colour, then the conversation has a common theme. If these joining parts form a (smaller or larger) circle, then the extent of conversation between the two neighbours is symmetric (they express their views to the same extent). However, if the joining part is formed by a larger and a smaller half-circles, than the conversation is dominated by one of the neighbours. As we can easily observe, the visualization of these conversations by Poly-Universe circles strictly follows and represents the given story and its various aspects.

4 CONCLUSION

The potential of Poly-Universe in studying and teaching information visualization, a crucial topic in nowadays scientific and everyday life, has been discussed in this paper.

Two different educational scenarios have been presented in this manner. These scenarios are suitable to develop the sense and need of information visualization, when Poly-Universe constructions are more than l'art-pour-l'art forms and when shapes, colours and positions have a well-defined meaning and various consequences can be drawn from the interpretation of the construction. It is needless to say that this method can deeply connect various disciplines to each other, such as Geography, Literature, Sociology, Political Sciences and Mathematics. What is also of great importance, that students can learn and understand symmetric and asymmetric relations in a contextual form: (geometric) symmetry can represent mutual understanding of each other, mutually advantageous commercial agreements between countries or companies, joint movements of equal parts. Besides the usefulness of the presented method in the education of practical aspects of information visualization, this deeper understanding of symmetry may even be more important from educational, cultural and scientific points of view.

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MULTIDISCIPLINARY SYMMETRY EDUCATION WITH POLY-UNIVERSE TOOLKIT IN SCHOOLS AND INFORMAL LEARNING CONTEXT IN FINLAND

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Abstract: *Between 2017 and 2019, Experience Workshop Global STEAM Network coordinated the Poly-Universe in School Education (PUSE) Erasmus+ project in Finland. In this article, we describe an example of a symmetry education school project, which was carried out in a Finnish lower secondary school with Poly-Universe Toolkit. In addition, the article provides a brief summary of the applicability of Poly-Universe Toolkit in a Finnish elementary, lower and upper secondary school context, based on the experiences we have gained in the PUSE project. Finally, we introduce Poly-Universe Toolkit in informal educational activities provided for adults in Finland.*

Keywords: Poly-Universe, STEAM, Symmetry Education, Formal Learning, Informal Learning

1 INTRODUCTION

Between 2017 and 2019 Experience Workshop (www.experienceworkshop.org) coordinated the Poly-Universe in School Education (PUSE) Erasmus+ project[1] in Finland. Members of Experience Workshop already have a broad history in implementing the Poly-Universe Toolkit (PT) for symmetry education in mathematics and the arts. The Erasmus+ project has given us the opportunity to deepen our knowledge, think further and conduct several experiments in classroom implementation together with Finnish teachers and students, and adapt the PUSE-methodology to the Finnish National Core Curriculum (FNCC, 2014). In addition to focusing on the classroom-based development of various PT-related tasks and projects, we created professional development projects for teachers and introduced the PT at several informal learning events. In this article, we summarize two types of examples of implementing PT for symmetry education in different contexts.

In the next part of this article, we describe an example of a school project carried out in a Finnish lower secondary school with PT. The third part of the article provides a brief summary of the applicability of PT in a Finnish elementary, lower and upper secondary school context, based on the experiences we have gained in the PUSE project. The fourth part of the article is introducing PT in informal educational activities provided for adults.

2 THE POLYUNIVERSE TOOLKIT IN FINNISH SCHOOLS: THE POLY-UNIVERSAL FRACTAL TREE

In Viitaniemi School in Jyväskylä, Central-Finland, a teacher of mathematics, chemistry and English, Merja Sinnemäki, and an art teacher, Leena Kuorikoski have been running the PUSE project. From the beginning of the 2018 Spring Semester to the end of the 2019 Spring Semester 7-9th grade students from Viitaniemi School have realized various math-art learning projects. PT was regularly in use for the playful approach of several topics in their mathematics curriculum with a special emphasis on the unique

[1] Erasmus+ PUSE Study (Poly-Universe in School Education) 2017−1−HU01−KA201−035938 Project.

opportunities which PT offers in symmetry education. During the PUSE project, students did playful problem-solving mathematics exercises with PT related to the Pythagorean theorem, calculation of percentages, fractions, area, perimeter and angles of various shapes, as well as PT's basic shapes: triangle, circle, and square. The 7th-9th graders also played a wide variety of combinatorics-based games with PT during the whole length of the PUSE project.

According to the feedback received from the teachers, we found that PT is an inspiring symmetry education tool. PT helped teachers to motivate and engage their students in various learning activities. It happened several times that the students requested using PT, and they tried implementing a newly elaborated learning topic as a PT-based game. PT provided opportunities for the students to realize multidisciplinary symmetry education projects. This type of working not only increased students' creativity but also promoted and enhanced positive interaction in the classroom between students.

The most inspiring opportunities were connected to combining phenomenon-based learning with symmetry education in the classroom and finding new connections between art, mathematics and science with the help of the PT. The most remarkable example was, in this regard, the Poly-Universe Fractal Tree Project. In this project, students combined their interests in fractal-structures with discovering PT's scale-shifting symmetry-related properties (PUSE, 2019) and creatively merged a STEAM activity for making a spatial fractal model, suggested by Huylebrouck (2014) and geometrically re-constructed several PT-modules. As part of the activities, they were also collecting information on several Chemistry and Biology topics included in their formal curriculum, such as photosynthesis and patterns in nature.

From a pedagogical point of view, the Poly-Universe Fractal Tree Project supported students to gain positive experiences about collaborative problem-solving across school subjects as part of a creative and active learning process. Both the learning content and the activities had several links to Symmetry Studies.

In the introductory part of the Poly-Universe Fractal Tree Project, students participated in small workshop activities in Chemistry and Biology. They have used PT to build artistic models of molecular structures and creatively modelled the photosynthesis process. They have studied the self-similarity characteristics of fractal-structures and compared it with the scale-shifting symmetry property of the PT modules. The triangular PT-module has been used to create a Sierpinski-triangle-like fractal model

(Figure 1) and students explored the phenomenon of scale symmetry, based on their creation. At the same time, the square-like PT-module has been used to experience various connections between the notions of composition and balance in art, and natural phenomena have been studied to discuss the notions of symmetry and beauty, both in geometrical art and in natural patterns.

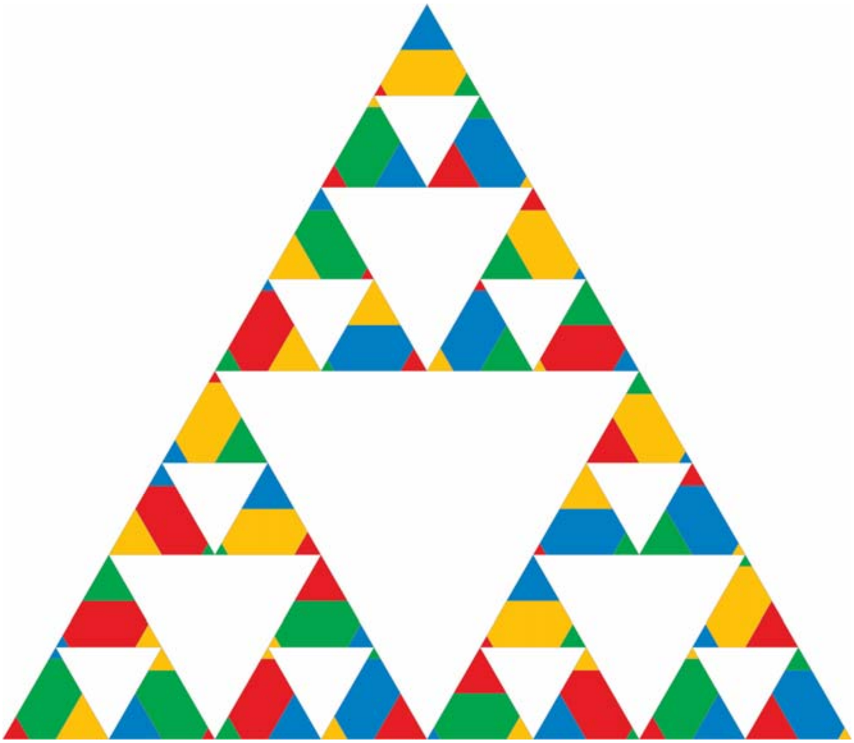

Figure 1: Sierpinski triangle constructed with PT. Student experiment in a Finnish classroom.

After the introductory activities summarized above, students were given physical materials and handcraft tools, such as cardboard, paints, rulers, knives, etc. and based on the instructions provided by Huylebrouck (2014), they started to figure out in small groups, how to build a fractal tree. After the successful brainstorming and discovering various exciting properties of 'spatial' fractals, the students made plans for the organization and distribution of the work and divided themselves into groups. Some measured the branches, some cut the cardboard and some painted the trunk and branches. After they started to work, the fractal tree was soon ready (Figure 2). Teachers were mainly observers of the progress. In the next round of activities, the fractal tree ’grew fruit’ by geometrically re-constructing several PT-modules, which were hung on the fractal tree by the students as ‘fruits’.

Figure 2: The Poly-Universe Fractal Tree Project introduced by Merja Sinnemäki (left) and Leena Kuorikoski (right) teachers at the PUSE Multiplier Event organized at LUMA Days by Experience Workshop STEAM Network.

According to the students’ feedback, they found the workshop interesting and fun. They thought that a workshop like this one would be a perfect activity for other students too, at the start or at the end of a curricular topic related to proportions and symmetry. They also thought that implementing the STEAM approach is a good way to support teamwork skills and positive attitudes toward learning-together.

3 THE POLY-UNIVERSE TOOLKIT’S POTENTIAL IN FINNISH ELEMENTARY, LOWER AND UPPER SECONDARY SCHOOLS

Based on the interactions with Finnish teachers, during and since the realization of the Poly-Universe in School Education (PUSE) Erasmus+ project, and according to the requirements emphasized in the Finnish National Core Curriculum and included in the

PUSE Methodology (PUSE, 2019), we can summarize PT's applicability in Finnish basic education, as follows:

PT has a unique potential to introduce and establish the practice of "hands-on" mathematics or learning-by-doing within a mathematics class (cf. Perusopetuksen, 2014: p. 128, 234, and 374.) because it supports: actions to enjoy "doing" the mathematics; the ability to consider different points of view in problem-solving and discussing mathematical topics; making mathematics touchable, less abstract, to understand the importance of applied knowledge (cf. Perusopetuksen, 2014: p. 236.); doing mathematics by involving the students' actions based more fully on their cognitive potentials, develop their motoric and spatial skills (cf. Perusopetuksen, 2014: p. 129.), sense of direction and location (the need to use their problem-solving, hands-on and community skills all at the same time); learning mathematics through active playing.

PT has unique potentials to develop skills in the geometric construction (cf. Perusopetuksen, 2014: p. 376.) and scientific categorization of different shapes and objects based on measurements and constructive problem-solving (cf. Perusopetuksen, 2014: p. 236.). PT can make playful and exciting of discovering the geometry of: the circle; the square; the triangle.

PT demonstrates the potential for developing mathematical/computational thinking by implementing strategic/heuristic approaches in problem-solving. PT offers more hands-on, and trial-and-error-based or open problem-solving paths than it is usually in formal textbooks by implementing conceptual thinking about: mathematical concepts (circle, numbers, square, etc.) combined through activities; making connections between concepts and visualizations; connections between mathematics and the arts by implementing procedural thinking in symbolic procedures.

PT has a unique potential in multidisciplinary/STEAM learning (symmetry, mathematics and arts) + e.g. introducing mathematical concepts in a foreign language. PT has been successfully implemented in learning mathematical concepts in English, when language learning is embedded in non-verbal/hands-on activities; discoveries in Mathematics and Arts connections and various aspects of symmetry in the following topics: Geometry & Measurement (grades 1-12); Combinatorics & Probability (grades 1-12); Sets & Logic (grades 1-12); Graphs & Algorithms (grades 1-12); Complex Problems & Visuality (grades 1-12).

Artistic connections become important in learning by implementing PT in: making own art, symmetry-related compositions out of geometric shapes; thinking about mathematical equations in visual ways; studying art and symmetry to discover mathematical and scientific notions; discovering the role of colours and shapes in symmetry; enhancing the visual impact of mathematics classes; making mathematical concepts come to life through imagination and creative activities.

School implementation of PT for children of ages 6-18 is supported by: easy-to-follow instructions on how to use; an easy-to-follow collection of problems for students and teachers; the online instructions and collection of problems is available in English for free; the online materials are easy to print. PT supports various learning styles, both on the individual and group level. PT is applicable even for children of kindergarten age to become familiar with basic shapes and geometrical aesthetics and basic notions of symmetry through simple games.

PT can support learning-together activities also in an extra-curricular and an out-of-school learning context.

The material of PT is very durable. PT is not only an educational tool but also a fun game, it can support a positive attitude towards learning and can increase motivation and engagement (cf. Lukion…, 2014: p. 130.).

4 THE POLY-UNIVERSE TOOLKIT IN AN INFORMAL EDUCATIONAL CONTEXT IN FINLAND: JOYFUL COMBINATORICS

“The Poly-Universe, in fact, is not simply a tangible tool or a closed-system puzzle, but an open math-art system which models the Universe, allowing a glimpse into its internal laws.” (PUSE, 2019: p.12.) Experience Workshop’s member, Kerry Osborne is a teacher who, when teaching in the Bronx, in New York City, worked extensively with Lincoln Center’s aesthetic education program; thus Zsuzsa Dárdai’s words of encouragement to open the mind to continue to tinker with this tool, she found to be a wonderful invitation. Osborne has been exploring ways the Poly-Universe circular discs lend themselves nicely to ‘ice-breaker’ informal educational activities. She has been tailoring ways they can assist in forming small groups of participants who, from the very start of a workshop experience, can look at a subject from multiple perspectives.

Haaga-Helia University of Applied Sciences in Helsinki (Finland), offers vocational education teacher training programmes taught through a mixture of hands-on physically present contact days and remote-learning connectivist methods. Each contact day begins with a warm-up activity. In one such warm-up, course participants entered the room and saw the circular PT discs placed on the floor - one ring of the same base colour with connections of the same size and colour. Stemming from that, a near ring with the same base colour and then branches using the remaining two base colours turn-about to focus on connections to one another via the same size and colour matches on the peripheries (Figures 3, 4, 5).

Figures 3, 4, 5: Kerry Osborne's informal PT workshop at Haaga-Helia University of Applied Sciences in Helsinki.

The circular nature of the discs championed the group to circle around the formation, and when asked, ‘What do you see?’ ‘Circles,’ ‘colours’ and ‘connections’ were among the responses.

This class, as it is taught in English, has attendees from nationalities which span the globe: Japan, Azerbaijan and Nepal to name a few, and the PT was used to hasten heartened sharing of educational backgrounds. Once prompted to think of a joyful memory attached to their childhood schooling experience, which one of the pieces reminds them of, they then were to pick up that piece and find a group of 3-5 people (using PT’s periphery connections to determine ‘belonging’ to a group) and share their memory within their group.

The participants were also asked to think:

- from what level of schooling is the memory – from primary or secondary schooling?
- where (in which country and locale – rural or urban) the school was?
- was it publicly funded or was it a private school?
- was the school co-educational or was it a ‘girls’ or ‘boys’ school?

As an example, Osborne, a facilitator of this activity gave after picking up a large yellow disc with green as the largest semi-circle: ‘From my primary years, or what in the USA is called elementary school years, I remember doing a lot of learning through games in Grade 2. In one board game we played, no one ever clambered ‘to be’ the yellow and green pieces, which made me so happy, for I was always able ‘to be’ my favourite colours. This was a publicly funded school which had both girls and boys and was located 16 miles to the west of New York City in the town of Montclair in New Jersey.’

The Haaga-Helia programme contains a teaching practice component, which Osborne is working to fulfil with Folkhälsan – a not-for-profit social and health care organisation that works, via education and research, to promote good health and quality of life in the Swedish-speaking areas of Finland. When conducting workshops for Folkhälsan’s Family and Relationships, and Nylands department meetings, Osborne has used modified versions of ‘joyful combinatorics’.

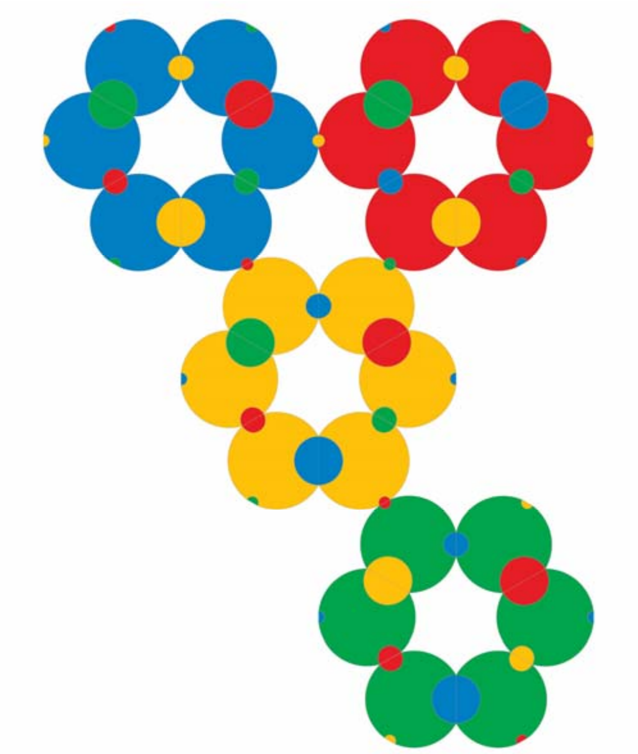

Figure 6: PT discs placed on the floor, using the same base colour with connections of the same size and colour.

Prior to the Family and Relationships (Famore) workshop, the tables and chairs were placed to the perimeter of the room and the PT discs were placed on the floor in four rings- each using the same base colour with connections of the same size and colour (Figure 6). This time the only prompt being to: ‘Choose a piece which appeals to you, and be able to share why it does. Then find three other people you can connect with to form a group of four by looking at perimeter matches. Explain, within your group, your reasoning for your choice.’

In Famore’s many programmes to improve well-being, being able to tune into another person, and build structures to support one another when experiencing both delight and dilemma are of key importance. As physical movement is an often-used tool in Folkhälsan’s well-being programmes, the ‘joyful combinatorics’ grouping activity transitioned into symmetry-based movement activity.

From each grouping of four, participants split into two pairs. They participated in partner symmetry mirroring exercises; however, rather than having the focus on only their partner, a participant could focus on the discs as well, and thus open themselves up to another in perhaps a safer bearing.

Being at ease with mathematical concepts and using them as tools to be at ease with others in a group – in an embodied manner, was explored further in the Nylands' department workshop led by Osborne. Rather than having the discs pre-arranged, they were merely scattered on the floor. When given the same instructions as the Famore group, there were natural clarification questions asked as to what sort of a connection should be looked for – and thus, the opportunity to clarify with 'perimeter connections with two alike semicircles'.

Participants of this activity can find it enlightening and fascinating when making a 'connection' and listening to another's choice. The reason for one connection's abundance e.g. of green being for a desire for nature and growth, and minimised blue -- due to the feeling of 'blue, of course, being such a cold colour'; however, a warm sunny yellow connection gave a window into seeing their group member's large swathe of blue representing 'the big blue, wide open Finnish summer sky, inviting you to get out and explore for hours on end.' (Figure 7)

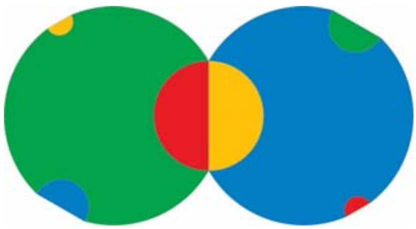

Figure 7: PT discs with the "big blue" on the left.

The Nylands' group also transitioned into body movement-based symmetry work. Guided to focus on mirror symmetry, the group explored first mirroring while holding the discs, and then decided to place the discs down and only have one another to mirror. Some preferred the first way, and some the second. Again, creating spaces through maths movements to feel safe in justification of one's point of view.

The use of the Poly-Universe discs for 'joyful combinatorics' self-sorting into small groups was then used by two Folkhälsan Foundation workshop participants, one from

Famore and one from Nyland, when they conducted an anti-racism workshop in Tampere for teachers. This kind of self-sorting and listening to others' points of view before tackling how to find solutions to difficult problems, was well received and facilitated the creation of groups keen to listen to various ways of understanding.

5 CONCLUSIONS

When imbued with PT inspired playful learning, project-based activities can lead to new perspectives on the characteristics of mathematics. Thus, organically alluring structures of mathematics, present both in nature and the built environment are revealed in elegant ways. PT works invitingly and guides the whole group to appreciate the beauty of mathematics for annotating thought.

Through group activities with PT, we can additionally practice using what mathematics has to offer as a tool for discovering the merit of others' thoughts. Rather than competing with one another for quick set answers, the PT modules lend themselves openly to using mathematics as a tool for making new discoveries. With such tools, a more robust ability to take into account varying views can occur from the beginning of understanding difficult problems, and this could guide the way to finding robust solutions; and perhaps even further questions, which are worth asking.

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POLY-UNIVERSE (PU) BEYOND POLY-UNIVERSE IN SCHOOL EDUCATION (PUSE)

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Abstract: *PUSE is the acronym for Poly-Universe in School Education which is, in turn, the name or title of an Erasmus + project*[1]*, funded by the European Union, which proceeded for two years under the leadership of János Szász Saxon and Zsuzsa Dárdai in the educational development of Poly-Universe, with the collaboration of schools and educational institutions in Slovakia, Spain, Finland and Hungary. The result of this project can be found at the following link: http://poly-universe.com/*

PU is a universe and as such goes beyond PUSE and should be considered as a mathematical environment in its own right where research can provide new and unknown ways to do and teach Mathematics. In this article we will try to discover some of these paths in the second of the aforementioned areas, that is, the teaching of Mathematics, but we do it beyond PUSE without losing its reference that has been vital for all of us by offering us the possibility of getting to know a new universe, with all that this entails.

Keywords: Mathematics education, 97 Axx (MSC 2010)

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[1] Erasmus+ PUSE Study (Poly-Universe in School Education) 2017−1−HU01−KA201−035938 Project.

1 PU IS A UNIVERSE

A universe is a set of elements that relate to each other following certain rules. A universe is more than a set, in that the latter can be understood as a mere gathering of elements. There is even, because of the need for operability, the notion of the empty set, whereas it would not make sense to speak about an empty universe because if there are no elements there can be no rules that unite them.

Poly-Universe is a universe formed by 72 pieces, which take the form of three well-known figures: the triangle, the square, and the circle. Therefore, we have to understand that Poly-universe's own name already describes what we are talking about. The 72 elements that make up this universe are organized, in addition to the aforementioned forms, by a partition of each figure into four parts that keep a proportional relationship between its sides (1/2; 1/4; 1/8) and areas (1/4; 1/16; 1/64) in a ratio of 1/2 that is repeated on each scale as it descends from the bigger parts to the smaller ones.

There are many mathematical universes, notably the universes formed by numbers among which we can mention those that have been the object of special study throughout the history of the development of mathematical knowledge: natural, integer, rational, irrational, imaginary, etc. and others that while not so elementary are part of the mathematical knowledge of most people: even numbers, odd numbers, palindromic numbers, decimals, unit fractions, proper fractions, and many more. It should be noted that it is the arithmetic operations and their development that created the need to expand the first and primitive universe that of natural numbers, to create new universes. Thus, in order to give any pair of natural numbers a subtraction, the negative numbers were created that together with the natural ones created a new universe: the integers. The same happens with the division of integers and the appearance of fractions and then of rational numbers, and so on. What makes us interested in the ontogenesis of numbers is to be able to observe how arithmetic operations are the main source of the dynamism of mathematics and are the generators of new universes. One of the greatest human creations has, of course, being the possibility of writing quantities far beyond Archimedes' sandbox using only two 10 digits and arithmetic operations, that is, the decimal positional numbering system. In other words, with 10 elements and basic operations we can create new ones that are certainly not infinite, but their finite number is not determined.

PU is a universe, it is a finite universe, just like the one formed by the 10 digits of the decimal number system, which initially has 72 elements. Just like with the operative (arithmetic) combination of the 10 figures we can represent "any" known finite quantity, we can likewise obtain new forms from the geometric and logical "combination" of PU pieces, that is, new inhabitants of this universe. For this, it is necessary to define these operations and in that PU is also versatile because we have the freedom to define them. We will talk about these operations that give life to PU.

2 POLY-UNIVERSE AND CODIFICATION

Coding, that is, labelling a series of elements so that they are easily recognizable, sortable, and recoverable is currently one of the most useful logical operations. Information retrieval is based on the assignment of one or several labels to an element, which allows its identification, classification and recovery. Among the various educational uses that we have been able to give PU, the creation and use of a coding system seem to be useful for the purposes indicated above.

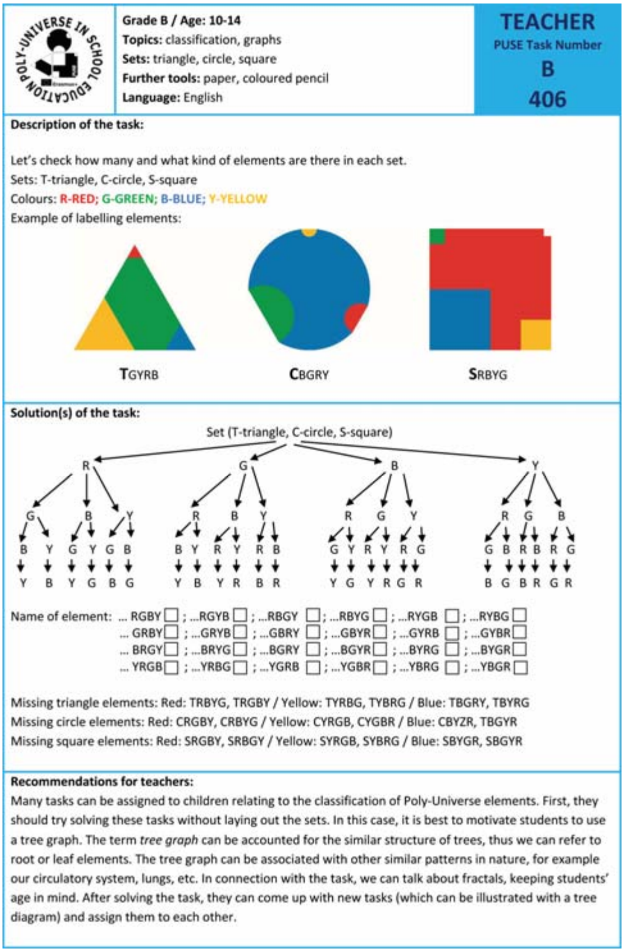
The coding system used was as follows:

The name of each piece consists of 5 letters.

The first is the initial of the form: S for Square, C for Circle and T for Triangle.

Below are the initials of the colours being placed in decreasing order of the areas of their fields. R for red, B, for blue, G for green and Y for yellow. Always in capital letters.

The following tree represents the coding of the PU pieces and is included in Task 406 of the PUSE book.



Grade B / Age: 10-14
Topics: classification, graphs
Sets: triangle, circle, square
Further tools: paper, coloured pencil
Language: English

TEACHER
PUSE Task Number
B
406

Description of the task:

Let's check how many and what kind of elements are there in each set.
Sets: T-triangle, C-circle, S-square
Colours: R-RED; G-GREEN; B-BLUE; Y-YELLOW
Example of labelling elements:

TGYRB CBGRY SRBYG

Solution(s) of the task:

Set (T-triangle, C-circle, S-square)

Name of element: ... RGBY☐ ; ...RGYB☐ ; ...RBGY ☐; ...RBYG☐ ; ...RYGB ☐; ...RYBG☐
... GRBY☐ ; ...GRYB☐ ; ...GBRY ☐; ...GBYR☐ ; ...GYRB ☐; ...GYBR☐
... BRGY☐ ; ...BRYG☐ ; ...BGRY ☐; ...BGYR☐ ; ...BYRG ☐; ...BYGR☐
... YRGB☐ ; ...YRBG☐ ; ...YGRB ☐; ...YGBR☐ ; ...YBRG ☐; ...YBGR☐

Missing triangle elements: Red: TRBYG, TRGBY / Yellow: TYRBG, TYBRG / Blue: TBGRY, TBYRG
Missing circle elements: Red: CRGBY, CRBYG / Yellow: CYRGB, CYGBR / Blue: CBYZR, TBGYR
Missing square elements: Red: SRGBY, SRBGY / Yellow: SYRGB, SYBRG / Blue: SBYGR, SBGYR

Recommendations for teachers:
Many tasks can be assigned to children relating to the classification of Poly-Universe elements. First, they should try solving these tasks without laying out the sets. In this case, it is best to motivate students to use a tree graph. The term *tree graph* can be accounted for the similar structure of trees, thus we can refer to root or leaf elements. The tree graph can be associated with other similar patterns in nature, for example our circulatory system, lungs, etc. In connection with the task, we can talk about fractals, keeping students' age in mind. After solving the task, they can come up with new tasks (which can be illustrated with a tree diagram) and assign them to each other.

Figure 1

To use this tree, the initial letter indicating the shape: C: Circle, S: Square or T: Triangle must be placed on the top. Following the branches of this tree, you get to build the code that corresponds to each piece.

Thus CRGBY is the name of the circle in which red is the base colour, blue is the colour of the largest field in the corner, the medium-sized one is green, and the smallest one is yellow. This codification is biunivocal, each name corresponds to a piece and each piece to a name. So the piece

Figure 2

corresponds to the name CRBYG

And the name CRYGB corresponds to the piece.

Figure 3

This coding allows us to identify the pieces quickly, as well as to classify them and to look for the missing pieces in a given subgroup of PU. It is biunivocal because each piece corresponds to a different name and vice versa.

3 EDUCATIONAL APPLICATIONS OF PU

3.1 Coding and its algebraic value

As we have already indicated, PU is a universe almost totally unknown because we have barely peeked into it. In the following paragraphs you can find some ideas that could be used to learn Mathematics in the exploration of PU.

3.1.1 Reading of graphs and diagrams

First of all, the understanding of this coding system and its representation through tree diagrams is of interest, like the one depicted in figure 1 and included in the PUSE book in Task 406 B. The reading of the diagram and the identification of the combinations that occur with the pieces of the game is very instructive for the students of Primary Education (A: 6-10 years; B: 10-14 years in the PUSE classification). It is a type of activity that can be used at the beginning of the introduction of combinatorics and probability since the use of this diagram allows a systematic count of all combinatorial possibilities of forms and colours together with the space they occupy in the element. These types of tasks were proposed in the development of the PUSE project and were well received among students and teachers who tried them. The transposition of this principle into other similar situations, in which the combinatorial possibilities must be calculated was relatively simple.

3.1.2 Search for parts and assignment of combinatorial rules

Assigning a name to each piece allows a search for the pieces, a search that can follow a logical-algebraic path, which is complementary to, but independent of the visual geometry. In this way, we can develop algebraic methods of systematic search that are complementary to the geometric vision and therefore have their own mathematical value.

So if we want to put together all the circle elements of the red base colour, and we have the pieces CRBYG, CRBGY, CRYBG, CRYGB

CRBYG	CRBGY	CRYGB	CRYBG

Figure 4

and we want to look for the missing ones without resorting to their intuitive vision, we could act following an algebraic method as follows:

If we write the codes of the elements that we have in the following table (information written in bold), we can deduce the information that is in italics, taking into account that each colour has to appear twice in each column with the exception of the red one that is present in each element.

In column "2" there is no G, therefore, we have to put it twice, in column "3" there is a "Y" so we have to add one more, and the same applies for "B". In column "4" this same situation is repeated. Once the columns are completed, the codes can be constructed and the figures searched. As you can see this type of task can be done without geometric representation using only coding, although here we added geometric representation to make this process more understandable.

Piece	1	2	3	4
CRBYG	R	B (1)	Y(1)	G (1)
CRBGY	R	B (2)	G (1)	Y (1)
CRYGB	R	Y (1)	G (2)	B (1)
CRYBG	R	Y (2)	B (1)	G (2)
CRGYB	R	G (1)	Y (2)	B (2)
CRGBY	R	G (2)	B (2)	Y (2)

Table I

To find the missing pieces in an algebraic way, it is enough to know that with the exception of red that remains the base (background) colour, the rest of the colours appear twice in each position. This is a general rule for any set of elements with a given shape and given base colour. Other rules could be sought for different cases and the search and application of those rules have great educational value in the management of codes and labels.

3.1.3 Creation of other coding systems

The coding system used in PUSE is only one among many other possible ones. An exercise of high educational value would result from proposing that a new coding system be invented. A combinatorial system based on finite elements can assign a code or label to each figure in a biunivocal way. The simplest would probably be to replace letters by figures by creating a coding system of the following type:

- The name of each piece consists of figures.

- The first figure corresponds to the form according to the code: 1 for Square, 2 for Circle and 3 for Triangle.

- Then the codes of the colours are indicated in decreasing order of the areas of their fields: 1 for Red, 2 for Blue, 3 for Green, 4 for Yellow.

- The CGRBY piece would be renamed 23124.

Algebraic rules would take new forms but their mathematical background would be the same.

3.2 Symmetry in PU

The symmetry of the pieces

The 72 pieces of PU are different, in the sense that there are no pieces that are mirror-symmetrical to others. But if one of them is turned around we will have the symmetrical one in relation to the chosen axis of rotation. In fact, the axis does not matter because by rotation the piece can be turned in the position of that obtained by other symmetries. One of the characteristics of PU is that a piece and its image cannot be seen at the same time if we have a single universe and set of pieces. Is this an advantage or an inconvenience? At first, it seems a disadvantage and one might think that PU actually contains 144 pieces, it would be enough to join two universes and rotate all the pieces of one of the sets to obtain 144 pieces that are geometrically different, whether in shape, colour or the rotation in which the different parts of a piece are arranged.

A figure and its image are distinguished by the following property: in one case, the different parts are arranged in decreasing order according to their area, we must follow an order of rotation and in its image, the order is just the opposite. That is to say, in one case it is necessary to turn to the right and in the other to the left.

We will analyse the case of squares.

Figure 5

This piece (SRBGY), oriented by setting its largest part horizontally on the left side, has its second largest part to its right (turn to the right to occupy that spot) and the next one on top (turn to the left to occupy that spot).

Flipping this piece would result in the following:

Figure 6

In this image, it can be seen that the directions of rotation indicated above are reversed. Although the name of this piece has not changed, its geometric properties are different. Therefore, geometrically they are different.

We could add one last code I (Right); E (Left) to distinguish them.

SRGBYI / SRGBE and generalizing PUI pieces on the side in which the aforementioned turn is to the right and PUE in which that turn is to the left.

The four forms that SRBGYI can take by rotation of 90, 180 and 270 degrees (in the negative direction/clockwise) are:

Figure 7

The four forms that SRBGYE can take by rotation of 90, 180 and 270 degrees (in the negative direction/clockwise) are:

Figure 8

The same can be repeated in relation to any other axis of symmetry that is chosen. Each square has only one symmetrical figure (image) that is the one that results from flipping it, the rest coincide by rotation. Discovering this geometric property of PU squares is of educational interest because it naturally relates symmetry and rotation, two of the fundamental movements in the teaching of Geometry.

The fact that they cannot be seen at the same time, unless two PU universes are joined, or a digital version of PU is available that allows copying and creating the image (FLIP), forces us to imagine it and therefore to mentally represent the piece once flipped. Imagining the piece once turned is a very interesting didactic exercise because it is a mental representation of its properties: shape, colour distribution and direction of rotation between the two major parts.

3.3 PU and mirrors

One way to visualize the symmetrical figures (images) of the pieces I (that is, in which the rotation between larger parts (2. and 3. Colour) is to the right) is to use mirrors. If we place a mirror vertically in front of a PUI piece we will have a reflected PUE piece. Imagining and visualizing to see if our imagination has worked correctly are two didactic exercises of great interest.

Another possibility to work on issues related to symmetry with PU is to place a PU figure in front of a pair of mirrors that together form a right angle. In the case of the square, three other reflected figures appear and it is interesting to imagine and draw how that new image will turn out and then see which of those pieces are I and which are E.

Figure 9 represents this situation.

Of these four pieces, there are two on each side, two PUI and two PUE, those located diagonally are the same pieces and on the same side.

3.4 The symmetry of compositions

Let us finish with a couple of problems.

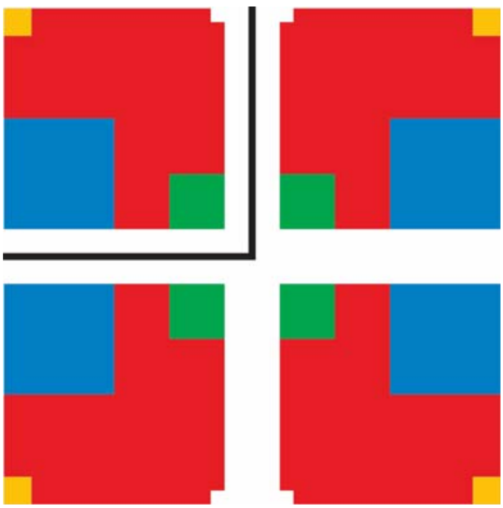


Figure 9

We define the composition of figures in the squares as the union of two, or more of them, sharing one side. Example:

Figure 10

This is certainly not the only way to define the composition of two squares but it gives rise to an operation that allows new figures to be created following a certain rule. It remains to be investigated and it is not the object of this article. If starting from a square, for example, the one on the left in a certain position of the four that we can choose by turning, by placing the other figure on one of the four sides in the four possible situations per turn the 4x4 = 16 compositions that arise are different or not.

Then the 3 other possible compositions without moving the initial piece by joining the second on the right side.

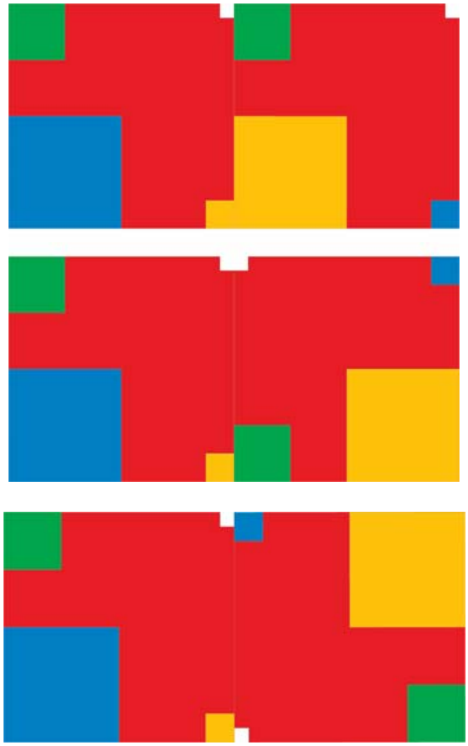


Figure 11

Problem 1: If a single PU is used (this is the case that we will analyse in this article) symmetrical figures in shape and colour cannot be created. This involves creating new ways of understanding symmetry. We will consider that a composition is symmetrical if colours are distributed symmetrically regardless of the shapes. The top figure is not symmetrical if we accept this definition. Can we make a symmetrical composition with these two pieces in relation to the colour distribution? Here is a simple and interesting problem. What movements will be necessary to do it? Is this a general rule?

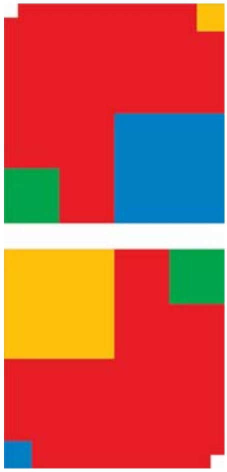


Figure 12

Problem 2: Is this composition symmetrical? In relation to what axis and criteria? How many different pieces do you need to compose it?

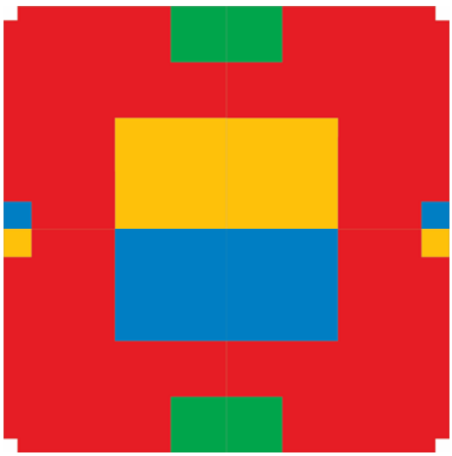


Figure 13

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OPERATION AND SYMMETRY OF THE POLY-UNIVERSE CIRCLE

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Books: PUSE Methodology (Poly-Universe in School Education 2019); 50 years – 50 Sculptures (Pauker Collection, 2019); Experience-centered Approach and Visuality in the Education of Mathematics and Physics (2012); Poly-Universe of Saxon (2010); Géza Perneczky: The Polydimensional Fields of Saxon-Szász (2002); SAXON: Dimension crayon (Espace de l'Art Concrete, 2000); MADI art periodical (No1 – No10).

***Abstract**: The "POLY-UNIVERSE" game, in using and developing the basic geometric shapes of János Szász SAXON's poly-dimensional plain painting, communicates a new artistic perspective to both nursery and primary school children and adults. Having a direct, by-touch connection with the geometric shapes, the sense of vision and of touch are developing and through the recognition of correlations and finding the linkage points, the ability of thinking improves and the skill of abstraction evolves. The Game does not only aim at problem-solving or recognizing colours or shapes, or solving logical puzzles, but also offers the possibility of playing a game freely, so children or adults can learn indirectly, through a game, an activity. When dealing with the different-scale basic geometric shapes and primary colours, they gain experience, discover and see correlations, points of linkage, then shape connections and the symmetry, while not knowing that they are learning. They can explore Poly-Universe, the realms of mathematics, art and philosophy, wandering engrossed in them, without being aware where they are. This novel game does not only develop skills or offer a visual-aesthetic experience, but it also expands the scientific knowledge, since it is based on an extraordinary mathematical set of proportions: Scale-shifting symmetry…*

Keywords: *Saxon, ArtGeometry, MADI, Symmetry, Poly-Dimensional, Poly-Universe, Game, Circle, PUSE, Combinatorics.*

1 OPERATION OF THE POLY-UNIVERSE CIRCLE

1.1 The structure of the basic elements

The Poly-Universe Circle is the linking of a basic object derived from a bigger circle and three smaller, similar semicircles: a big, a medium and a small one. The semicircles are situated at a 120° angle to one another, and their radius ratio is 8:4:2:1. A colour - red, yellow, green, blue - is assigned to each of the elements. Therefore the number of the elements in each package presents itself through the simple permutation without repetition of the four colours in all of the three cases, i.e. each package contains 4 • 3 • 2 • 1= 4! = 24 elements.

Figure 1: Design of the basic object, linking big, medium and small semicircles, a circle basic element.

1.2 Placing the basic elements

The basic elements can be placed in basic position and inverse (mirrored) position, and when placing, they can be turned in any direction in the plane. This is an aesthetic and technological consideration, but it has some mathematical consequences too. The orientation changes at the axial mirroring, thus we get a new element regarding the setup opportunities when we mirror, or else when we turn round an element.

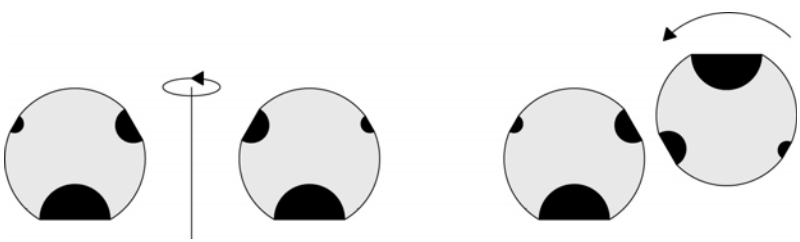
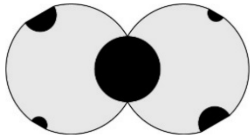
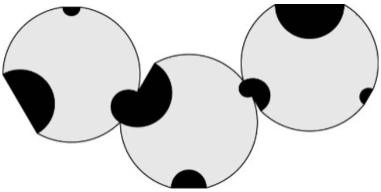


Figure 2: Basic position, inverse (mirrored) position, turning in the plane.

1.3 Linkage points

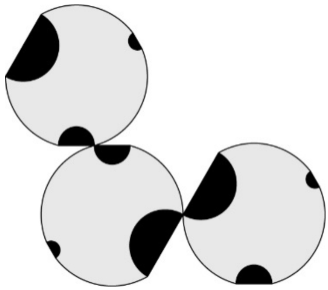
1.3.1 The basic elements can be placed with entire circle joins; then the vertices of same-size semicircles are connected with each other, thus creating smaller regular circles.





1.3.2 The basic elements can be slid along the diagonal of the semicircles, that is, we connect the vertex of a bigger and a smaller semicircle with each other.

1.3.3 Finally we can safely work with one vertex of a semicircle connected with that of another one on the plane.



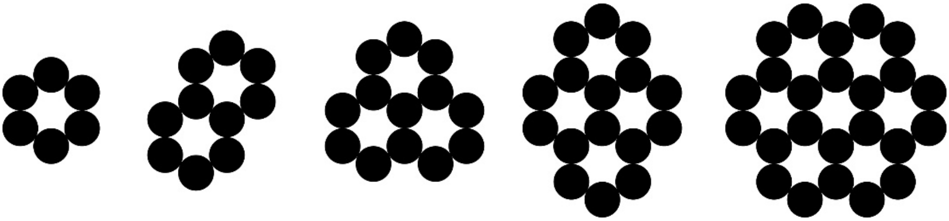
1.4 The rules

Let us find semicircles of the same size and connect them.
Let us find semicircles of the same size and of the same colour and connect them.
Let us find semicircles of different size but of the same colour and connect them.
Let us find semicircles of the same size and of different colour and connect them.
Let us find semicircles of different size and of different colour and connect them.
Etc...

1.5 Suggested methods of creation

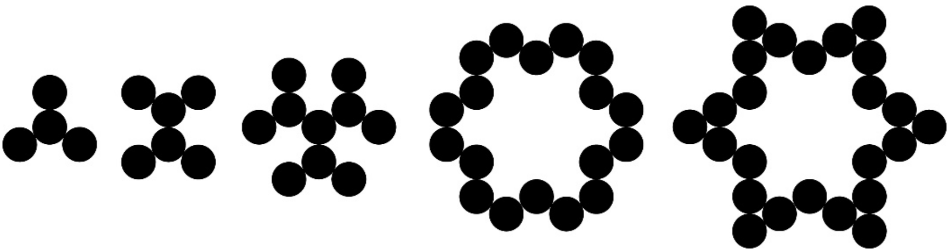
In the following part, we can learn about the different possibilities of methods of creation suggested by the inventor. We can create closed configurations, regular rings or open (irregular) configurations as well. We can work along the linkage points with entire circle joins, sliding and connecting the vertices of the semicircles, in the following ways:

1.5.1 The objective is to create closed rings with entire circle joins, in the first step using the fewest, then gradually all the 24 basic elements. Let us start following the examples below and if possible, let us use the rules separately:



Elements: 6 10 13 16 24

1.5.2 The objective is to create open or irregular configurations with entire circle joins. First, we use an optional number of elements, then all the 24 basic elements. Let us start following the examples below and if possible, let us use the rules separately:



Elements: 4 6 10 18 24

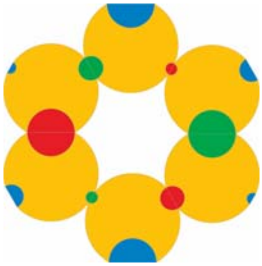
1.5.3 The objective is to create any open or irregular configurations, or even closed rings by sliding the semicircles. First, we use an optional number of elements, then all the 24 basic elements. Let us start following the examples below and if possible, let us use the rules separately.

***Summary**: Either we use the closed or the open method of creation during the game, we should always aim at creating the most regular configurations possible using all the basic elements at one time. The - practically infinite - number of mathematical and aesthetical possibilities inherent in the Poly-Universe toy only manifest when you apply all the combinations of scales and colours at the same time. Step by step we can get to creating and then solving the hardest problems – thus the beauty and harmony in the operation of the Universe will be revealed to us through the game.*

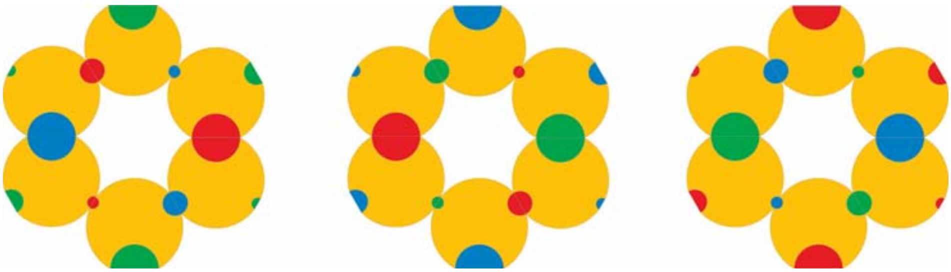
2 SYMMETRY OF THE POLY-UNIVERSE CIRCLE

2.1 In case of the same basic colour and of the same size and colour connection

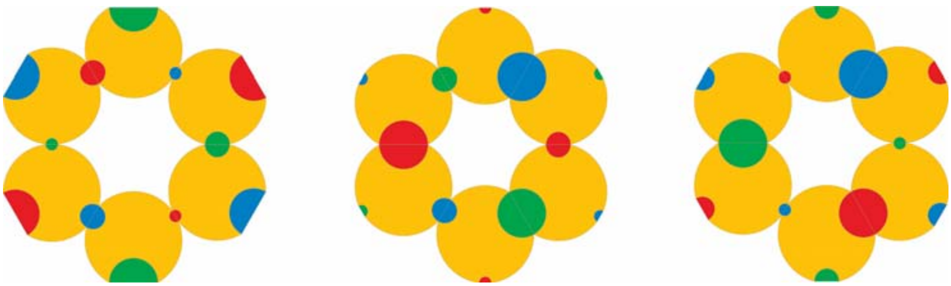
We choose elements of the same base colour from the Poly-Universe basic element circles, then construct a closed ring shape with same colour and size connections (see PUSE Combinatorics 225C task).



The question is how many different rings can be constructed this way? Are there line-symmetric shapes among these? Are there rotational symmetric shapes?
It is useful to group the possible constructions based on either the connecting coloured shapes or the non-connecting shapes on the periphery of the rings. The 6 possible solutions can be seen below:



Shapes in the first row are neither line- nor rotational symmetric. If we ignore colours, they have rotational symmetry of 180° (central point symmetry).



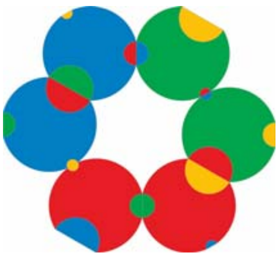
Shapes in the second row have three lines of symmetry, rotational symmetry of 120° and 240° if we ignore colours and only focus on shapes and sizes.

Recommendations: *It is worth recording the solutions we've already found: drawing, taking a photo, or constructing them from the basic elements using a computer. It is crucial not to look for solutions with trial and error but group solutions systematically.*

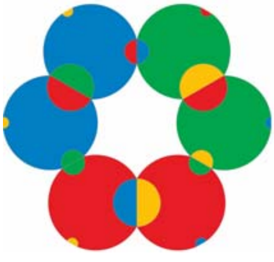
2.2 In case of the different base colour and of the same size connection

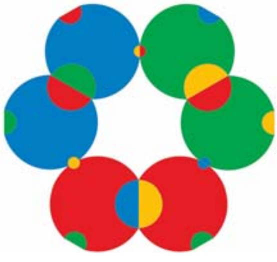
Examining the symmetry of the rings we can see that, we must pay attention to the fact that constraints on building the rings are due to the arrangement of colours (not to the geometry of shapes). When observing the connections of rings, the complete task can be developed by eliminating the rule of the same base colour connection. How can we answer the questions if we ignore the colours, concentrating on the sizes only? In that case, only the same size connections matter, we can discover further construction symmetries of shapes.

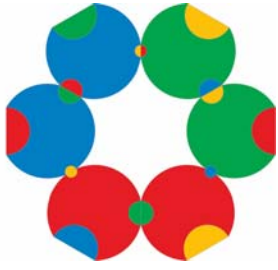
2.2.1 Rotational symmetry of 180° diagonal arrangement: semicircles of the same size are opposite each other and all elements are connected by a semicircle of the same size (in the ring), so one arrangement is possible



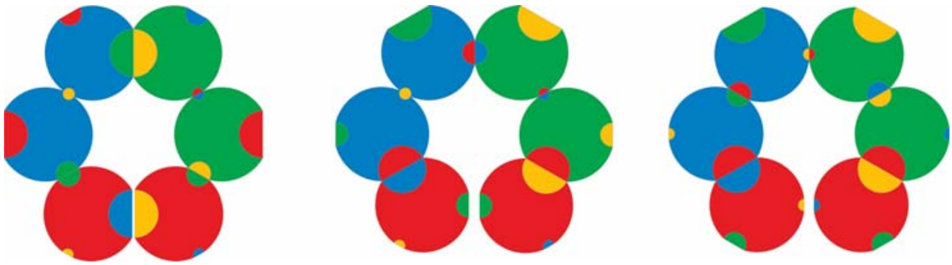
2.2.2 Rotational symmetry of 120° arrangement: semicircles of the same size are triangular with respect to each other, and two sizes are present at the same time. Solution 3 types of layouts are L + M, L + S, M + S.



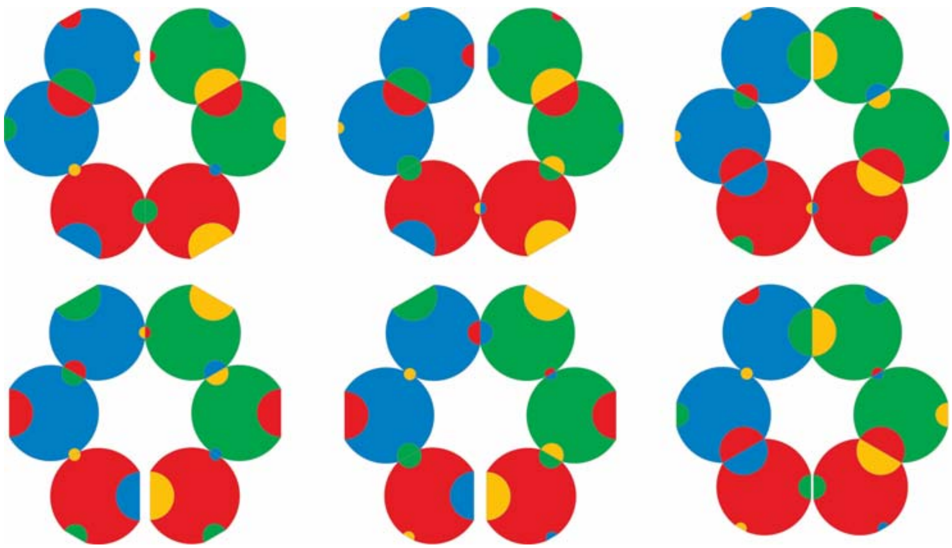




2.2.3 One possibility for reflectional or mirror symmetry: In the axis, all three smaller L, M, S semicircles are alternately present and each is duplicated. There are 3 types of such layouts.



2.2.4 A second possibility for reflectional or mirror symmetry: In the axes, all three smaller circles L, M, S are alternately present, one of them is 3 times, the other one twice, the third one only once, eg. 3 small, 2 medium, 1 large. With this mirror symmetry, 6 different layouts are possible, 3S + 2M + 1L; 3S + 2L + 1M - 3M + 2S + 1L; 3M + 2L + 1S - 3L + 2S + 1M; 3 L + 2M + 1S

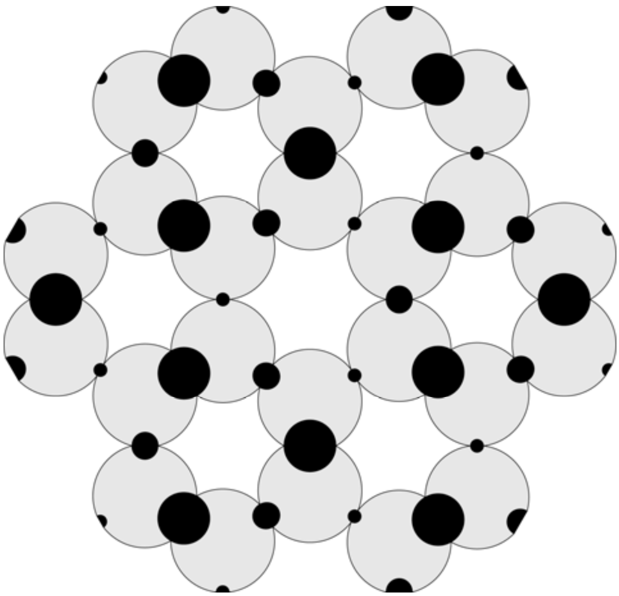


Summary: *When examining the inner symmetry of the circle arrangement, the total number of solutions is 13, that is, there are 13 ways to start laying the rings if we only consider the size of the smaller semicircles at the connections. In this case, the colour of the base elements is ignored, but smaller semicircles can still be connected by the same colour, while we can see the last joint will always be inaccurate, that is, we cannot close the ring.*

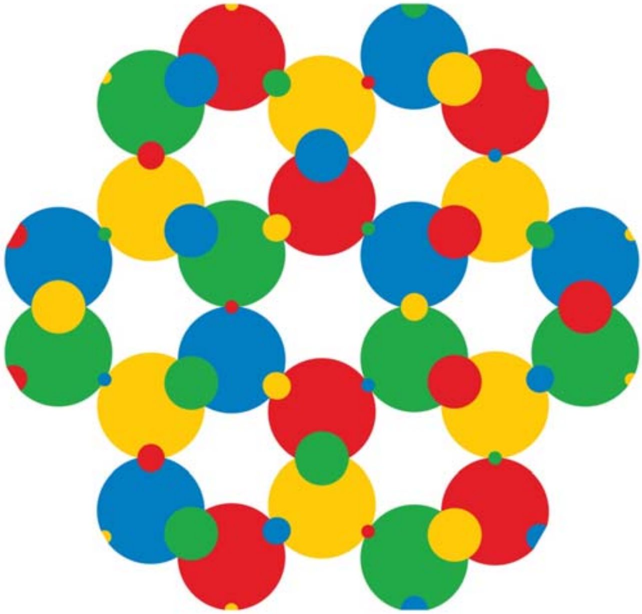
3 BUILDING THE RINGS OF THE POLY-UNIVERSE CIRCLE

In the previous task, we constructed rings of 6 basic element circles. In this chapter we move on and connect the rings with one another, paying attention to symmetries we found. If we ignore the base colours and only observed sizes when connecting the elements, we could make rings with two types of symmetries. Let's say that rings with point symmetry have symmetry type “one”, and rings with rotational symmetries of 120° and 240° (rings with 3 lines of symmetry) have symmetry type “two”.

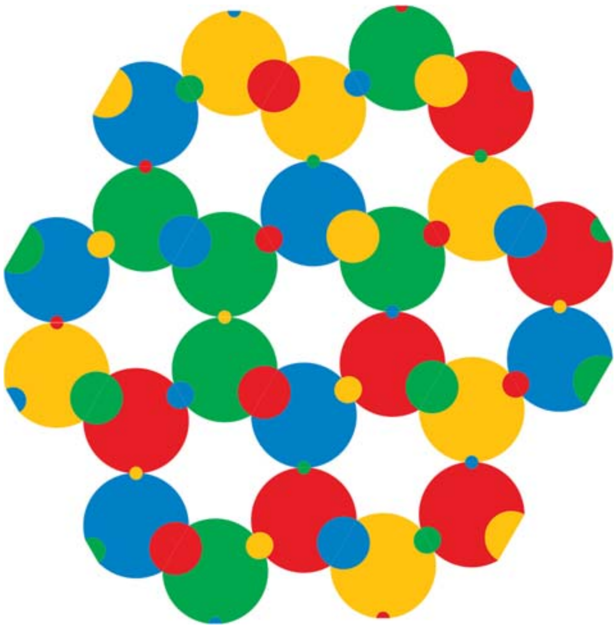
3.1 Now we construct a connected shape from the circle set with same size connections, connecting the rings. The question is: How many different connected shapes can we construct? What is the maximum number of rings we can connect from one set, with which kind of layout? It is relatively simple to construct a connected shape of 1, 2, 3, 4, 5, 6 rings with same size connection. The maximal shape which can be constructed from the 24-piece set without remaining elements is a symmetric shape of 7 rings, the “ring of rings” (see PUSE Combinatorics 228C task).



3.2 Now we construct a connected shape of 7 rings with the same colour and size connections. Constructing a symmetric shape of 7 rings from the 24-piece set without remaining elements is also possible. But due to connections of the same colour, this task is much more difficult, it requires at least an hour.

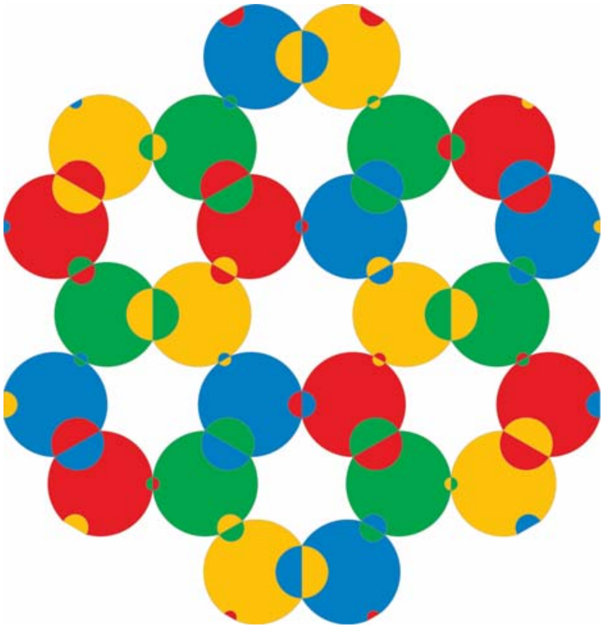


3.2.1 The question is: If we build a shape of the highest number of rings applying either type of one symmetry only ("one" or "two")? Which type of symmetry connections leads to a solution? We can see here too, there is a solution for both of them.



3.2.2 Another question: Is it possible to construct a shape of the highest number of rings from one set if both types of symmetries can be applied, and/or thus, we connect the elements randomly? Yes, with trials we mostly get this kind of solutions. But we can see the last joint will always be inaccurate, that is, we cannot close the ring. In this case, connections are not always precise due to the deformation of the rings (see Part of 2.2.3 and 2.2.4).

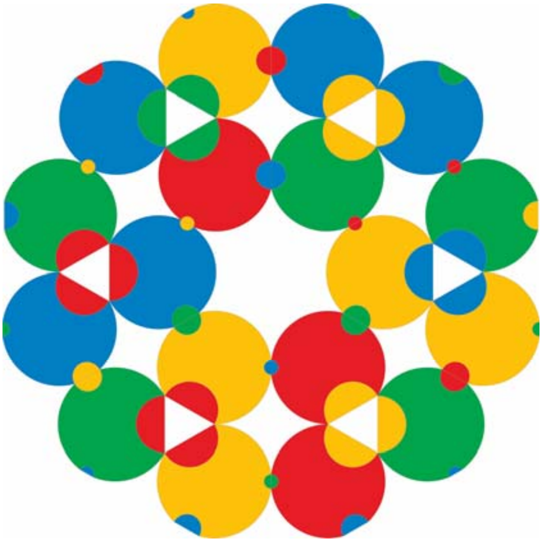
3.3 Finally we construct a connected shape of the highest number of rings from one set with complement connections (so, the base colour of the first element should be the same as the connecting semicircle of the second element). There is a solution, we've seen some layouts so far, and this task is much most difficult than before.



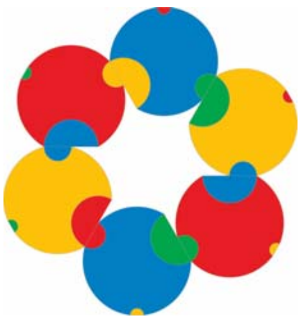
Recommendations: *During many years of testing, a few dozen solutions have been found for the construction of 7 connected rings with the same colour and size connections. To calculate the number of possible solutions of constructions with either type of symmetry and mixed type of symmetries alike, a comprehensive algorithm is needed.*

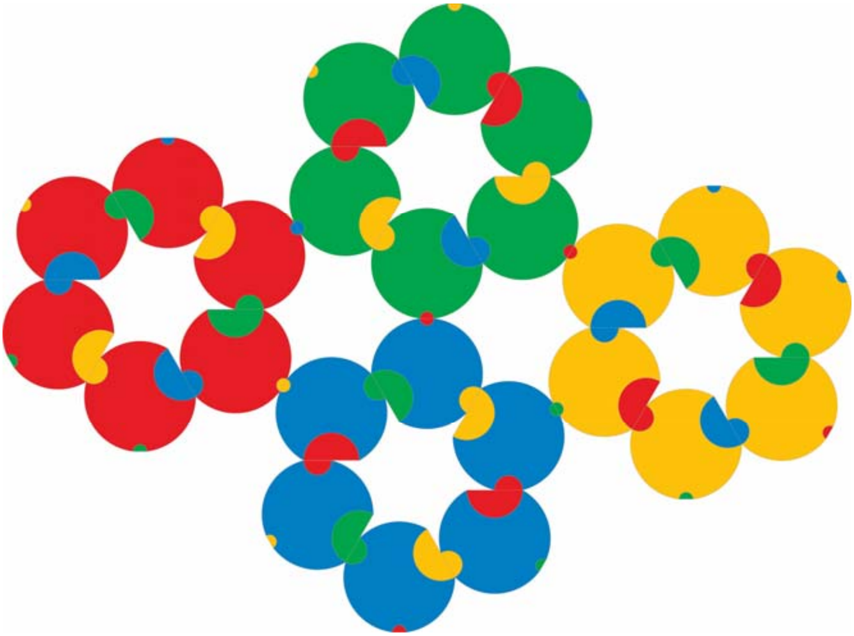
4 BUILDING THE SPECIAL SHAPES OF THE PU CIRCLE

4.1 We connect three basic element circles with the same colour and size connections of the large semicircles (see PUSE Combinatorics 230C task). The question is, how many shapes can we construct this way, using the entire set? Is it possible to connect small or medium-size semicircles in this layout (with vertex connections)? After constructing a possible layout of the entire set, can we rearrange it in a way that the base colour of the elements is different in each shape? How many different layouts are possible, using the entire set? Etc...



4.2 We connect the rings at six same base colour element circles, with the same colour and different size connections of the semicircles. The question is, how many shapes can we construct this way, using the entire set? Can we construct this way a symmetric shape of 7 rings, using the entire set? After constructing a possible layout of the entire set, can we rearrange it in a way that the base colour of the elements is different in each shape? How many different layouts are possible, using the entire set? Etc...





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FRIEZES, ROSETTES AND POLY-UNIVERSE

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Abstract: *From times immemorial, man has always been interested in symmetrical patterns and ornaments. Cave paintings, decorations of ancient ornaments; personal objects and artwork of tribal peoples are all evidence for this. Yet even more amusing to recognise not simply a symmetry but a symmetry group on a building or on a piece of textile. Friezes and rosettes – the topics of this essay – are the two simplest examples to symmetry groups. First, we give a general description of friezes and rosettes, then take a detour to folk art. Finally, we observe the possibilities of the Poly-Universe triangle and square elements from the perspective of friezes and rosettes. We are looking for tasks in the methodology book compiled for the PUSE project (Poly-Universe in School Education) that include friezes and rosettes. We examine how the task itself or the questions raised can be developed if we consider this perspective. Then we observe the possibilities, which frieze or rosette patterns can be constructed from the Poly-Universe. Is it possible to arrange all of them? Emphasizing the importance of the multidisciplinary approach is another aim for this essay. When starting off from the examination of any phenomena or problem, we always arrive at aspects of the world surrounding us. Now observations on the symmetries of the Poly-Universe set reminds us of many natural phenomena and works of art. Several examples to this are mentioned, and we also present our findings in detail, with pictures: frieze patterns of cross-stitch embroidery.*

Keywords: Poly-Universe, PUSE, frieze symmetry, rosette symmetry
MSC 2010: 00A66, 97U60

1 SYMMETRY, PLANE ISOMETRIES

Though symmetry is a wider concept, people mostly think of reflection when it is mentioned. Generally, we consider it symmetry if the given object is invariant to an operation (that is, some properties are preserved) (Darvas, 1999). Dissymmetry is a breaking of symmetry: the phenomenon, law or shape keeps the symmetry in its main concept, but not necessarily symmetric in its details. We can meet dissymmetry quite often when observing nature and art. Consider the human face or a panel of the coffered ceiling in a church. Asymmetry simply means the lack of symmetry. In this study, we cover plane isometries and frieze and rosette groups generated by them. The classification of the four plane isometries by direction preserving property and the existence of fixed points can be seen in Table 1.

	There is a fixed point	No fixed point
Preserves direction (motion)	Rotation	Translation
Reverses direction	Reflection	Glide reflection

Table 1: Classification of plane isometries

1.1 Symmetries in the PUSE methodology book

When studying symmetries in this essay we imply forms but ignore the colours. More precisely, if we say that, for instance, two figures can be mapped onto each other by translation, then after translation, the same forms but not necessarily the same colours will cover each other.

In section A of the geometry chapter in the book some tasks directly ask about isometry. In 110_A the task is to colour the reflected image of triangle, square, circle elements; in 102_A they build shapes that remind them of rotation (Figure 1). It is well worth observing the ratio of symmetric and non-symmetric shapes created by children during tasks allowing for free play and the flow of creation.

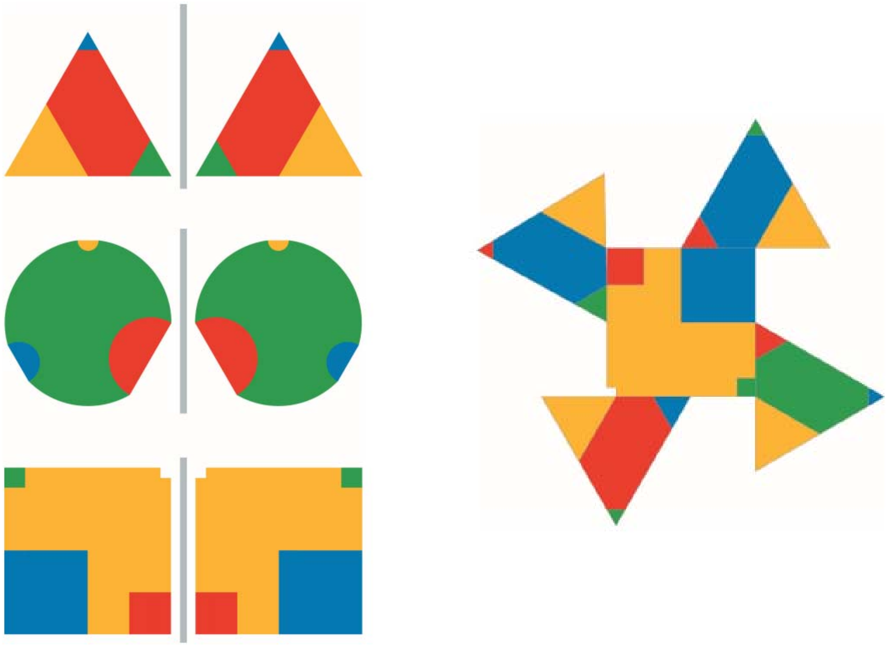

Figure 1: Figures for task sheets 110_A and 102_A.

Task 505_A is the following: "Work in pairs: both of you should construct a shape of 3-4 elements that your partner cannot see. Then show it to each other for a few seconds and cover them. After that draw the shape your partner constructed for you from memory. Check each other's solution if you are ready. If you succeeded, continue the game by constructing the given shapes using the elements of your own sets."

It is useful to ask about the symmetries of the constructed shapes. The teacher methodology sheets provide some of the most interesting symmetry constructions (Figure 2).

Apart from the top right picture, each construction seems to have reflection symmetry. Having a closer look at them, though, we notice that only the top left construction has perfect reflection symmetry, the other two just almost. What is the reason for this imperfect symmetry? The axis of symmetry is not where two shapes meet (as with the top left figure) but it crosses the figure – and none of the Poly-Universe elements is mirror symmetric.

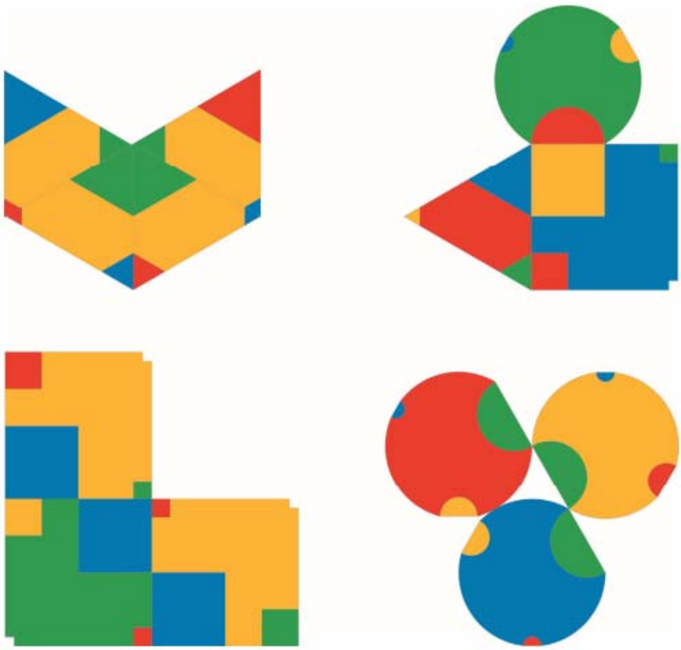

Figure 2: Figures for task sheet 505_AB

2 GENERALLY ABOUT FRIEZE SYMMETRIES

The class of isometry groups containing a single translation are *frieze groups*.

There are exactly seven ways of creating (infinite) linear patterns (friezes) which are generated by the (infinite) repetition of one motif. The codes and patterns of these subgroups (represented by letters) are shown in Table 2. The explanation for codes is the following:

Letter *p* at position 1 comes from the word pattern.

At position 2 there is a letter *m* if the pattern contains reflection symmetry with an axis perpendicular to the direction of translation. Otherwise, we write *1*.

At position 3 there are letters *m* or *a* if the pattern contains reflection symmetry with an axis parallel to the direction of translation, or a glide reflection, respectively.

At position 4 comes the order of rotation. It can be proven that frieze patterns can only have rotation centres of order two.

Group code	Pattern
p111	...LLL...
p112	...NNN…
p1m1	...DDD...
p1a1	...bpbpbp..
pm11	...AAA...
pmm2	...HHH...
pma2	...ΛVΛ...

Table 2: Classification of frieze symmetries.

Each of these frieze symmetries also appears in the Hungarian decorative art from the era of the Hungarian conquest of the Carpathian basin (Bérczi, S. 1986), but similarly can be found in other nations' art. We can learn about it in *Eurasian art* collection edited by Szaniszló Bérczi, which can be downloaded from this website:

http://www.federatio.org/tkte.html.We come across friezes everywhere: when walking along an old street if we look up to the ornamentation of the buildings – or look down, noticing the border of an ornate floor or panelling. Each of the seven frieze patterns appears on Hungarian cross-stitch embroidery as we read the study of István Hargittai and Györgyi Lengyel. (Hargittai I. & Lengyel, Gy. 2003).

2.1 Frieze symmetries on Hungarian cross-stitches

On a website (http://qtp.hu/xszemes/mn.php) I found a huge array of cross-stitch patterns selected from the collection of the Hungarian Museum of Ethnography. A question arose in my mind: would it be possible to discover all frieze symmetries among these? Well, I did, as Figures 3-7 show. Originally these patterns could be found on tablecloths, bedsheet, pillows, shirts, overlay patterns on traditional Hungarian coats. The most common patterns were the ones with only vertical reflection symmetry or the ones with vertical and horizontal reflection symmetries. The most difficult task was to find patterns without inner symmetry, or patterns which have reflection symmetry with a horizontal axis but no vertical axis.

Figure 3: pmm2
NM 8823
Pillow-end pattern, originally with long-armed cross stitch and cross-stitch embroidery
Time of creation: end of XIX. cent.
Place of creation: Diósad.

Figure 4: p112
NM 126854
Pillow-end pattern
Time of creation: the second part of the 19th century
Place of creation: Kalotaszeg (Kolozs county).

Figure 5: p1a1
NM 127238
Table-cloth. A mix of long-armed cross stitch and cross-stitch.
Time of creation: First quarter of the 20th century
Place of creation: Kalotaszeg (Kolozs county).

Figure 6: p111
NM 126917
Pillow-end
Time of creation: 2nd part of 19th century
Place of creation: Kalotaszeg.

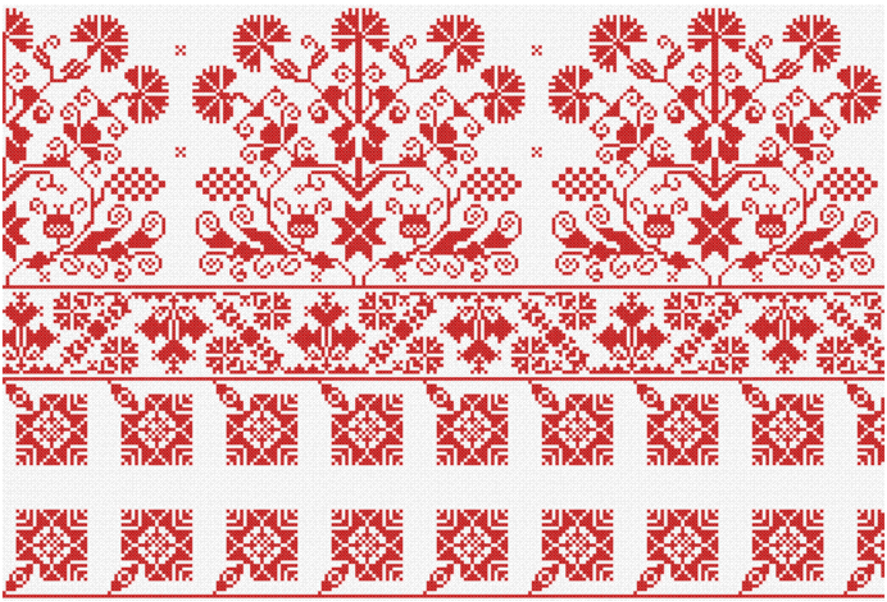

Figure 7: This cross-stitch shows 3 frieze symmetries: the upper pattern has pm11, the middle pma2, and the lower pattern p1m1 symmetry.
NM 51.14.655
Sheet-end, originally cross-stitch and long-armed cross-stitch embroidery.
Time of creation: turn of 18-19th century
Place of creation: Transdanubia

2.2 Friezes and the Poly-Universe

As we can see on figures 8-21, Poly-Universe constructions allow for all seven frieze symmetries. Those patterns which have reflection symmetry with an axis parallel to the direction of translation can only be assembled by arranging the elements at least into two lines (because Poly-Universe elements do not have inner isometries).

Figure 8: p111

Figure 9: p112

Figure 10: p1m1

Figure11: p1a1

Figure 12: pm11

Figure 13: pmm2

Figure 14: pma2

Now let's see which tasks of the book can be extended by questions on frieze symmetry.

Task 104_A asks to build quadrilaterals by assembling triangle elements. We can establish that an odd number of triangles make up trapezia, with an even number we get parallelograms. This task can be extended by being the following question: with connections of the same colour and size, what type of symmetry does the constructed linear pattern have? The answer: frieze pattern with p1a1 symmetry (Figure 11).

This extension may be applied to other tasks, e.g. 106_A, 130_BC. At 306_A, the task is to identify the rules of the connected triangle elements. Apart from the many rules, it is worth adding symmetry rules like p111 and pm11 frieze patterns.

Figure 15: p111

Figure 16: p112

Figure 17: p1m1

Figure 18: p1a1

Figure 19: pm11

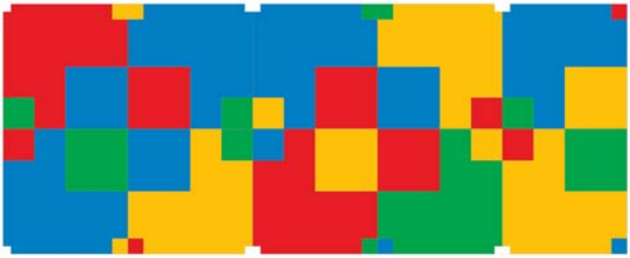

Figure 20: pmm2

Figure 21: pma2

Finally, the most beautiful frieze pattern from Poly-Universe elements in the book is the decorative frieze at the beginning of chapters, constructed by János Saxon Szász. This p1a1 frieze, in Figure 22, contains glide reflection.

Figure 22: p1a1 frieze, decorating the PUSE book.

3 GENERALLY ABOUT ROSETTES

The *Rosette (rose window)* group is a translation-free group, its name comes from church windows. There are infinitely many rosette groups which can be classified into two substantially different subgroups. One of them consists of groups of rotations about a single point: rotations with integer multiples of $2\pi/n$. They belong to the cyclic rosette group, written C_n. These groups do not have reflection symmetry. Besides the mentioned rotations, if the group also have *n* reflection symmetries with axes going through the centre of rotation, they belong to the dihedral group, written D_{2n}.

 

Figure 23[1]: Left St Peter and Paul church Gorlitz, C_6 rosette group. Right: Cambridge, Cambridgeshire, England, UK, photo by Leo Reynolds, D_{10} rosette group

Rosettes are not simply on buildings, we may find plenty of them in nature: when we cut an apple or orange in half; or when we marvel at a lovely flower or cactus, and we may even find starfish from the D_{10} symmetry group. Among old door locks and manhole covers, we can discover rosette symmetry; what's more, looking at our cars' hubcaps we can quickly find their "rosette code", and we could continue for long.

3. 1 Rosettes and the Poly-Universe

The book contains many examples to rosettes assembled by triangle and square elements. We can arrange 6 triangles to make up a hexagon with D_6 rosette symmetry. Examples of this can be seen on task sheets 125_B and 212_B (Figure 24).

Using the entire set, large hexagons assembled by triangle connections of the same colour and size have D_6 symmetry, see Tasks 130_BC and 212_B. The picture on the right in Figure 24 is an example of this. Task 212_B from the Combinatorics chapter asks for the number of constructions with the same size and colour connections. There are 12 ways to construct smaller (of 6 triangles) and larger (of 24 triangles) hexagons as well with the given conditions. We can establish that each construction has D_6 rosette symmetry.

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[1] Sources of pictures:
https://commons.wikimedia.org/wiki/File:St_Peter_and_Paul_church_Gorlitz_round_window.jpg
https://www.flickr.com/photos/lwr/6333291955/ (2020.02.06)

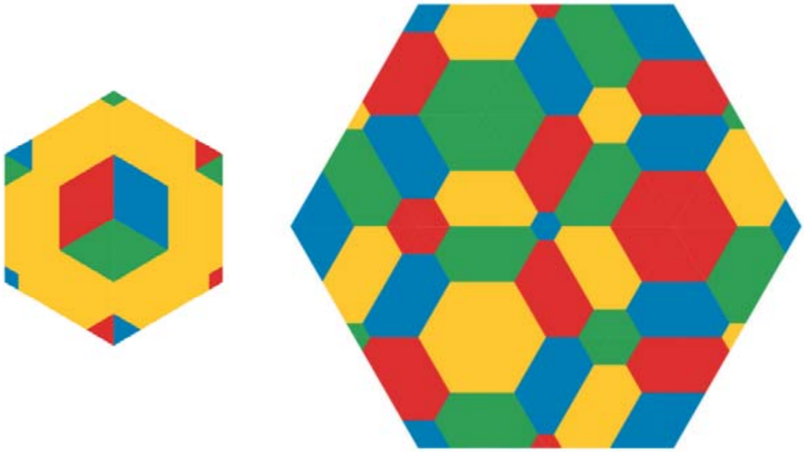
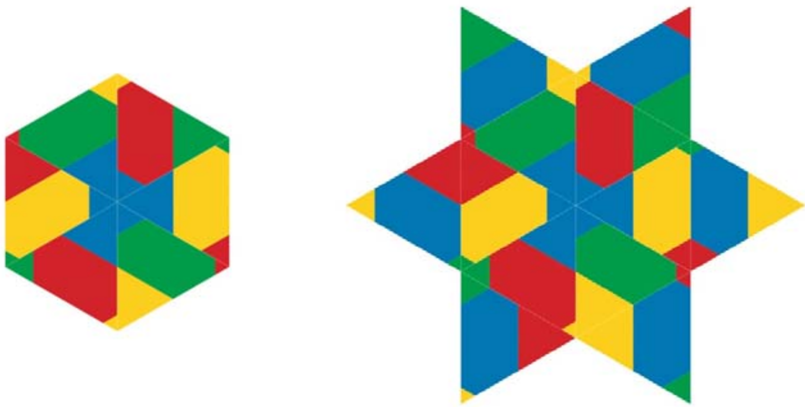


Figure 24: Rosettes from triangles on PUSE task sheets.

Flipping through the book we notice that the shapes we construct from triangles can either have no symmetries or have dihedral symmetry (see Figure 24). Would it be possible to construct rosettes with cyclic symmetry, using triangles? The answer is yes, but to keep the restriction of total side connection, we can only work with connections of the same colour and give up on same size connection.

See Figure 25 for examples, all three of them are C_3 cyclic rosette groups.



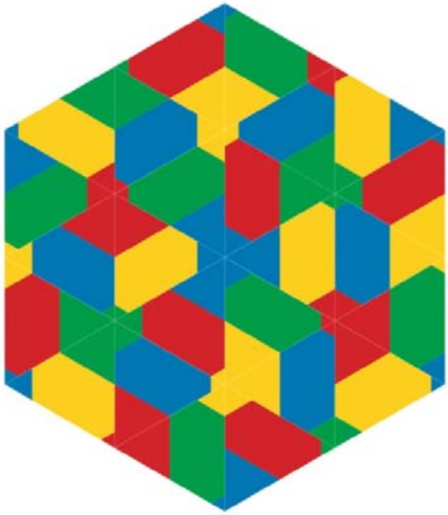


Figure 25: Cyclic rosette groups from triangle elements.

Nice rosettes can be constructed from square elements as well. First, let's examine the task sheets from this perspective. Many questions could be raised about symmetry in connection with Task 108_A; now the given solutions do not contain any symmetries at all. One possible solution of 231_C on the teacher sheet has D_4 dihedral symmetry (Figure 26).

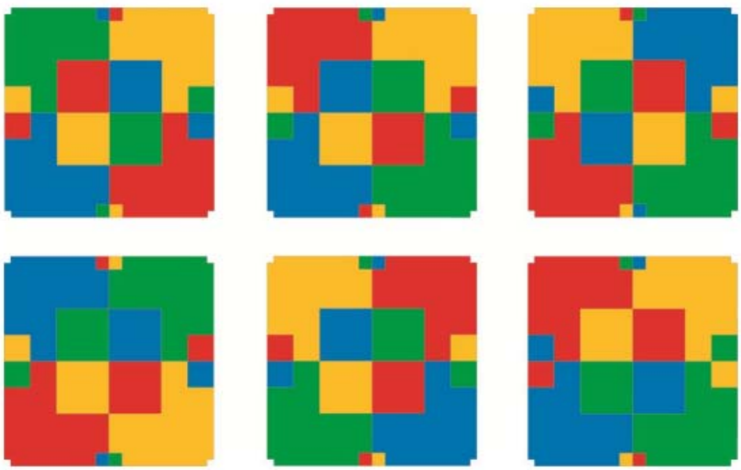


Figure 26: Rosettes from square elements on PUSE task sheets.

In the case of squares also, we can ask whether it is possible to construct a cyclic group. The answer is yes, two examples from Task 203_A represent the C_2 cyclic group (see Figure 27).

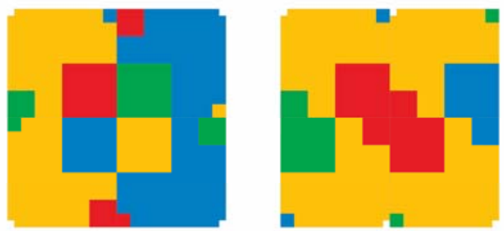


Figure 27: Cyclic rosette groups from square elements

We can also ask whether it is possible to construct bigger shapes (using more than four squares) of rosette symmetries.

Finally, on rosettes, we need to mention Task 508_AB which encourages group work constructions, using all three sets. For this, one solution on the teacher sheet is also a D_6 dihedral symmetry rosette.

4 SUMMARY AND PROSPECTS OF MOVING FORWARD

As we can see, Poly-Universe offers endless possibilities, and there are almost unlimited questions to raise. When we listed friezes and rosettes that can be constructed from Poly-Universe we did not aim to be exhaustive. We did not give an answer to the question asking for the number of ways to assemble each pattern; we simply wanted to show that there are suitable ways of combining the Poly-Universe triangle and square elements so that all frieze and rosette patterns can be constructed. We would like to encourage the Reader to look for further nice patterns and try to answer the proposed questions for the Poly-Universe circle elements, and also for mixed sets. Hopefully, we will return to these questions in a future study.

Figure 28: Rosette from mixed Poly-Universe elements.

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SEARCH FOR A COMPUTER AIDED SOLUTION OF THE 7×7 SQUARE POLY-UNIVERSE: AN ALGORITHMIC APPROACH

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Abstract: *Saxon's Poly-Universe was created to show that a logic-stimulating mathematical game could derive from an artistic concept. It combines colours with basic geometrical shapes on two-dimensional plane to set up three different board games (square, triangle and circle). In these games, players can customise the rules, the difficulty level of which increases with the compactness of the form, and the homogeneousness of the pattern preferred. We have chosen the square pack to test software algorithms capable of solving the most difficult player requirements possible. The purpose was also to optimise the computing time with the hardware capacity.*

Keywords: mathematics and art, science education, gamification, C# programming, combinatorial optimisation, test driven development, Poly-Universe, MADI art, János Saxon Szász

1 OUR HISTORY

About a decade ago we created some algorithms either to solve some non-trivial problems, or for visual entertainment purposes.

There was a recurring theme about these—computing capacity is limited, be it memory consumption or processing performance.

Problems with an intuitive human solution are often nigh impossible for a computer. Thus emerges the need for not just coding optimisation, but a logical evaluation of the solving algorithm to try and reduce its complexity (cf. Palócz and Katona, 2019).

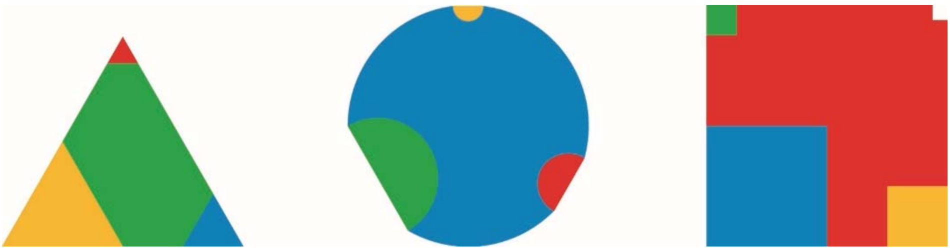

Figure 1: Basic elements of Poly-Universe: triangle, circle and square.
(Source: Saxon Szász, Stettner, *et al.*, 2019)

2 ABOUT THE GAME

Hungarian artist János Saxon Szász (Perneczky, 2002) created the first basic artwork in 1979 he called the 'Universe', and continued to carve and paint more complex MADI objects as modules of Poly-Universe till 2009. He chose three elemental geometrical shapes of the plane—the triangle, the circle and the square—each to which he added other inscribed shapes (Fig. 1). He drew smaller squares with different colours and sizes to the main square's corners. The same game applied to the triangle, and three different semi-circles were attached to the basic circle, both having a triangular pattern therefore. After producing three basic prototypes, Saxon Szász (2010, pp. 85–111) noticed that he might apply different combinations of the given colours to the same shapes. As a consequential result, 24-piece packs were worked out from every prototype (Fig. 2).

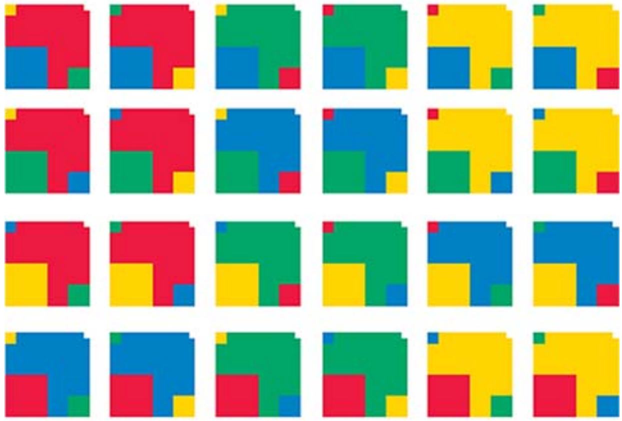

Figure 2: The 24-element square pack based on the possible colour combinations.
(Source: Saxon Szász, Stettner, *et al*., 2019)

The pieces could then be organised into greater free or tied forms with optional combinatorial logics, where players could choose among the possible rules. Most evidently, the shapes were the nexus between the pieces. For example, players could attach the identical shapes of the same or a different colour to each other in order to find interesting patterns. Aligned with an educational research (Saxon Szász, Stettner, *et al*., 2019, 407–414[1]; Darvas, 2019), a combinatorial quest evolved from this experiment, setting up two criteria:

1. Use all $n \times 24$ elements to create a greater form (n is a positive integer), and
2. Apply the same rules to all the elements used.

There are some which proved to be less or more feasible among the possible rules. Based on certain experiences without the use of computers, the difficulty of solutions increased with the compactness of the greater form and the homogeneousness of the corner patterns. The latter means that only identical colours and shapes meet at the corners. Our investigation focused on the possible solutions for the notably most difficult rules (the most compact form with the most homogeneous corner patterns).

[1] The numbers represent task sheets, not pages.

We have chosen the square pack also because it proved to be the hardest among the three to solve with the given rules. The positive integer of the first criterion was defined as $n = 2$, for it is easy to see that $n = 1$ does not provide any solution for the square pack. With these premises, we developed a software algorithm to comply with the following requests:

1. Use complete square pack with 48 elements ($n = 2$),
2. Choose the most compact form (7×7 square with one empty slot), and
3. Give the most homogeneous corner pattern (only identical shapes and colours meet).

Compared to the triangle or the circle, the peculiarity of the square pack is that one of the corner shapes are ‘missing’, which means it has no colour, and its tiny shape is cut out of the base square. The tiny hole may either be considered as if it had a ‘fifth colour’ that is always combined with the tiniest corner shape (see the square in Fig. 1). Other corner squares alternate between three different sizes and four different colours (red, yellow, green and blue) compared randomly, but with the condition that the same combinations may only be used *n* times (one full table pack contains 24 different elements).

3 THE TERMINOLOGY

While such a definition of the Poly-Universe keeps it within limited parameters (Saxon Szász, 2010), we understand it to be a game played on a board, placing cards from one’s hands following certain rules. In order to get into the concept, we need to develop a vocabulary. Our definitions are as follows.

Game: used to describe the entire activity.

Game State: denotes the status of the board and our hand at a given point in time.

Card: piece of equipment to be placed. In this game, cards are squares with four edges, each of a different size and colour, one of them a small cut.

Pack: a full collection of distinct cards, one of each possibilities.

Hand: a collection of cards.

Edge: part of the card that can be considered for matching purposes. Edges have a colour (or shape) as well as size. The base colour can be considered an edge, too, for the lack of a better word.

Joker Edge: an edge that can be matched with anything else.

Joker [Card]: a card consisting of joker edges.

Board: the area where cards are being placed.

Layout: the angle and position of the cards, relative to each other.

4 EARLY CONCEPTS

The very first approach has been to use Windows Excel to try to produce all possible board combinations of the $n = 1$ basic pack (24 pieces and 5×5 square board with one empty slot), and to simply validate them with the following rules:

1. Adjacent cards should have matching colours, and
2. No card shall be used twice.

In all cases, the row number would be split (using a combination of modulus and integer division).

The most basic solver tries all cards in all cells, which results in 24^24 variations (cf. Saxon Szász, 2010, p. 110). While its simplicity may appear promising, the combinations to validate are overwhelmingly numerous. One slight refinement ensures that cards cannot be reused, producing 24! (620448401733239439360000) variations—an 80 bit number (2^79.04).

This reduces the cases massively by removing one cause for invalidity; however, it also adds a significant amount of calculation: for every cell, previously used cards have to be excluded. However, this is also computationally burdensome, while still being restricted to a single pack. Two things became clear very quickly:

1. Excel cannot contain the required number of rows, and
2. It cannot handle numbers large enough to describe individual cases.

5 THE SECOND APPROACH

Some logical restrictions have been noted. First, the starting card can be fixed, as the game is symmetric. Colours can be considered as parameters (numbers or letters), so any solution found has 24 variations for all possible colour combinations.

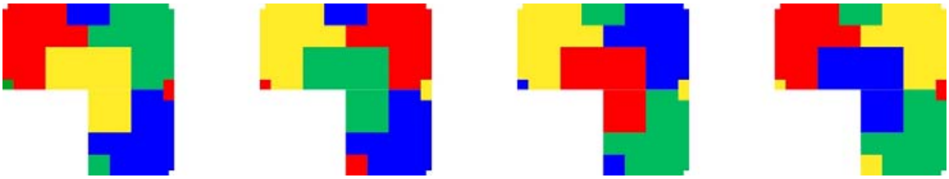

Figure 3: A combination of three-card sets from the 24-card pack, applying the colour-shape-match rule—fourth cards cannot be placed. (Drawing by the authors)

Secondly, a two-by-two cell with the shaped edges at its corners cannot be filled using a single pack (see Fig. 3). Once the first card is placed, the two cards adjacent to it are deterministic: two edges are fixed, the third is the one not yet used. This leads to a scenario where the fourth card needs to have the same colour at its diagonal edges. Hence a 5×5 board cannot be filled using a single pack.

Parallel to this, a new idea has been considered: since this game requires edges to match completely (colour and size, also meaning shape), it is possible to consider the connection points on the board to be solved (Fig. 4). This presented a last opportunity to try to produce all possible placements given their very limited nature. It still proved to be a dead end, stretching the limits of the tool with a small board.

6 THE C# APPROACH

We used PRO C# 7 multi-paradigm programming language for generating, testing and optimizing (Troelsen and Japikse, 2017). During the process, we considered some relevant 'mantras' mentioned by today's developers and learning sites[2]:

---

[2] See https://www.dofactory.com , or https://dzone.com .

Keep writing flexible, with extendible code. This usually means adding various abstractions to the code, at the expense of runtime and memory consumption, regarding additional objects to instantiate and interfaces to resolve (cf. Gibson and Senn, 1989).

Premature optimisation leads to unreadable code.

Do not implement everything in front, because the chance that *You Ain't Gonna Need It* (YAGNI) is very high (Wäyrynen, 2004).

Keep It Simple, Stupid (KISS). If there is a straightforward solution, prefer that over a complicated, fancy one (Kemerer, 1995).

+ R B G	R + G B	+ Y B R	Y + R B	+ R B Y	R + Y G	+ Y G B
B G + Y	G B Y +	B R + G	R B G +	B Y + R	Y G R +	G B + R
+ Y R B	* * * *	+ G Y R	G + R Y	+ R Y G	R + G Y	+ R Y B
R B + G	B Y G +	Y R + B	R Y B +	Y G + B	G Y B +	Y B + R
+ G B Y	G + Y R	+ B R Y	B + Y G	+ B G R	B + R G	+ R G B
B Y + G	Y R G +	R Y + B	Y G B +	G R + Y	R G Y +	G B + Y
+ G Y B	G + B R	+ B R G	B + G R	+ Y R G	Y + G R	+ Y R B

Figure 4: A possible solution of the two-pack board (7×7) provided by the test console. Each card has an identical pair, and all the 2×24 cards are consumed. Assuming that the shapes are fixed, we focused on the combination of colours (R: red, Y: yellow, G: green, B: blue, +: small cut, *: joker edge) that match at the edges. The top left corners of the boards are solved logically.

Depending on personal experience, such advice can seem contradictory. In any case, refactoring is possible.

Given our earlier experiences with brute force algorithms, we opted for a simple solution to try to increase our chances at an acceptable runtime and memory consumption. This means using simple data structures, such as integer arrays, and trying to extract that part of the model, which does not change while progressing to a solution. Where optimisation did not seem beneficial, we left the door open to improve testability and extendibility.

For our first attempt, we picked the easiest rule set: straightforward to implement and easy on resources. This rule set assumes:

A rectangle-shaped grid, where a few slots can be left empty (represented by a joker card).

Neighbouring edges must match their size and colour.

Specifically, trying to fill a 7×7 board with 2 packs of cards and a single joker (hole).

Other variations exist with an elevated level of possibilities based on the original concept of the inventor Saxon:

Open shape: the algorithm has to figure out where to put the cards, while avoiding arriving at identical partial solutions in multiple ways.

Shifted cards: instead of creating a textbook grid, cards could be aligned to match the size of the coloured edge.

Different colours: instead of matching colours, they can or must be different.

Different edges: instead of matching sizes, they can or must be different.

Going beyond edge matching and requiring distant edges to meet rules, possibly expressed otherwise: in each group of four, each colour/shape combination must appear exactly once.

5 colours (pack of 60 cards with the cut edge or 120 cards without).

We realised and utilised the symmetry of the board:

> Alignment: The card alignment is the same in all corners, meaning that the board implicitly contains its horizontally, vertically and diagonally mirrored solution. At the same time it is possible that we find multiple solutions which can be transformed into each other.
>
> Colours: once we find a solution, we can swap the colours around freely, which results in 24 variations.
>
> The algorithm does not use edge size explicitly, so any size variations (diagonal flipping) are valid.

In order to avoid identical duplicate partial solutions, the board is solved in a deterministic sequence. The main idea was to solve row by row, then column by column. It is slightly better, though, to solve the game in 2×2 blocks: once the first card is placed, the ones next to it have two edges defined, reducing the number of possible cases. Doing so, one must realize that jokers must be placed proactively, e.g., right away in the top left corner, instead of waiting to get stuck.

Starting with all 49 possible variations is not better—any cases with the joker in the last 3 rows (or columns) would be solved the same way, ultimately failing. A deterministic sequence, so that we do not get the same result, could be slightly optimized yet for jokers.

Other ideas came up like a deterministic first piece, and a fixed layout due to the rules. The latter helped with the shapes, since if they are known, we may focus on the colours. An easier rule set results in a fixed layout and the smallest number of possibilities, compared to the “different edge”, “different colour” or “non-deterministic layout” variations.

We also researched the possibility when we start with all the possible locations of the joker, and with the possibility of multiple jokers. Next, we noticed there is no solution with a joker at the edge. It means that we would test a lot of identical cases with the joker in the bottom line.

7 TESTING

Unit testing is a fad these days. We could not identify very many cases, but verifying the basic logic of the game was useful, resulting in important questions to answer:

Are the cards in the pack different?

Are there exactly two states in the first generation, placing either the joker or first card?

Does the single-pack, no-joker game run following its deterministic path?

Our focus has been on a (rather simple) presentation of game state and verifying its correctness. In addition to observing cases, we also added the ability to observe the progression of a single solution by keeping parent-child relationships. This allowed us to verify that cards are indeed placed in the correct order.

7.1 Case A: 2×2 board, 1 card pack, 1 joker

Result: There are four results, with the joker in each position. (Mathematically, these are mirrors of a single solution, the board being symmetric.)

Once the first non-joker card is placed, the next two cards are determined (two edges must match, and there is only one other card with the same two edges, the one with a different 3rd colour), eventually requiring the 4th card to have the same colour on the opposite edges (Fig. 5).

*	*	R	+
*	*	G	B
R	G	G	B
+	Y	Y	+

Figure 5: A single hand determined by a joker card in the top left corner (R: red, Y: yellow, G: green, B: blue, +: small cut, *: joker edge).

7.2 Case B: 2×2 board, 2 card pack, 0 jokers

Result: There are two results, with an additional partial solution with 3 cards (Fig. 6).

+	R	R	+
B	G	G	B
B	G	G	B
+	Y	Y	+
+	R	R	+
B	G	G	Y
B	G	G	Y
+	R	R	+

Figure 6: Two hands determined by the same starter card in the top left corner (R: red, Y: yellow, G: green, B: blue, +: small cut, *: joker edge).

The first card is given. There are two options to the right: the same card again, or the other one with the same two edges. Starting with two identical cards on the first row, there are only two (identical) cards remaining with the bottom colours, with the other colour. However, starting with two different cards, the bottom colour can only be the same as the top colour—red in this case—since the two cards must match and the other three colours are used.

8 RAMP IT UP

Once convinced that the program is correct, we gradually increased the board size to 7×7 (1 joker) and 8×6 (without). The program finds 2 solutions for the 7×7 case (see Fig. 4 and Fig. 7; cf. Saxon Szász, Stettner, *et al.*, 2019, 414), and 12 for the 8×6.

The 8×6 is much lighter, taking about 0.274 seconds, peaking at 6048 ways to place 25 cards. The 7×7 on the other hand takes 13.281 seconds, peaking at 289036 ways to place 30 cards (including the joker), consuming overall 2–4 GB memory in the process. Curiously enough, while there are 198702 ways to place 20 cards already, it drops to 62690 ways for 27 cards.

Having so many intermittent solutions inspired us to improve the mechanism to reduce the number of branches, hence the upgrade from a row-by-row to a 2×2 approach. While the change indeed reduced the number of intermediate solutions and thus memory consumption, the runtime did not change a bit (in the debugger) due to a very few extra mathematical operations - suggesting that minimising abstractions was probably a good idea.

For other possible use cases, we have to consider the freedom of rules. Assuming a progressed game, it might tell whether there is a solution, show any or all, and allow hints.

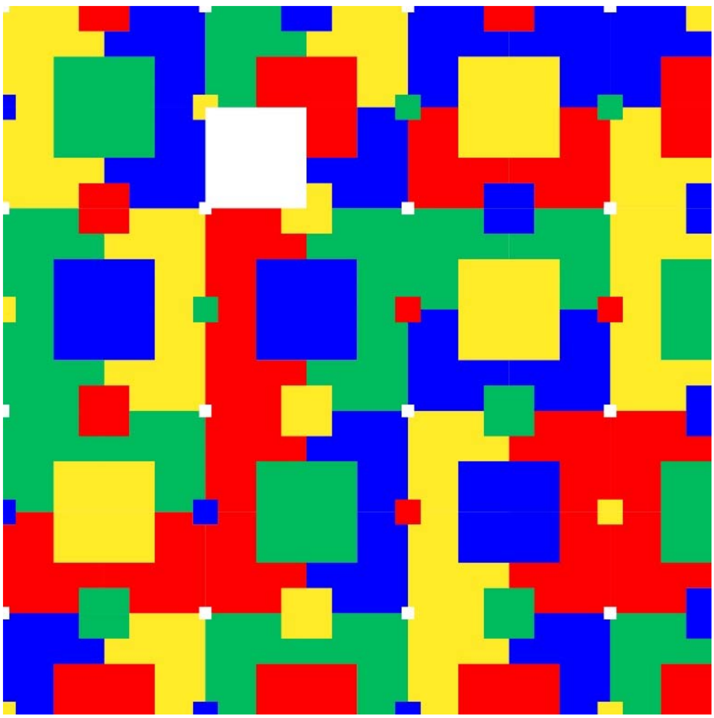

Figure 7: A possible solution of the two-pack board (7×7) provided by the test console. Each card has an identical pair, and all the 2×24 cards are consumed. Shapes and colours match at the edges. (Drawing by the authors)

Considering the resource requirements even for this most simple case, improvements are needed to make it viable for players, especially for rules giving more freedom. As achievements, we could save partial solutions. We did not use threads, which could improve runtime while hiding processing efficiency.

9 FURTHER PROSPECTS

Applying a new concept with the simple design (YAGNI) and premature optimisation, we got completely opposite results. The advanced method to find the next cell generated fewer cases with the same runtime.

For further extensions we need to decide if we go along with object-oriented or procedural solutions. We also need to decide if we should use a static container for method references or solver classes.

An exploration of multithreading would require to split the processing into two parts, which means the algorithm would find the next cell first before it fills it. But can we process two threads, or follow multiple queues? Are generations needed anyway?

If a rule set produces too many solutions, processing could be altered from horizontal to vertical (generations). After finding a result, the process should be stopped and saved. It is also possible that calculation follows a diagonal or Z path (in Excel: A1, B1, A2, B2, C1, C2, D1, D2...) on the board. If a joker is placed, the game should continue with another four pieces.

In each case, we should be able to save the actual state of the calculation process. It could improve the computer's capacity if we discard the partial and unsuccessful results. This diary may also help reconstruct threads, but it overtaxes the memory beyond a given amount. Overall, Saxon's Poly-Universe is a challenge not just to human thinking, but to modern computational science as well.

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HOW CAN QUADRILATERALS BE CONSTRUCTED FROM POLY-UNIVERSE TRIANGLES?

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Abstract: *In this study, we first get to know the significant properties of a regular, i.e., equilateral Poly-Universe triangle through a pedagogical approach, such as the geometry structure, the symmetry arrangement in the interior, the colour combination system and its coding, and so on. In the second part, we experiment with the construction of symmetric QUADRILATERALs from POLY-UNIVERSE triangles. First, we look at the isosceles trapezium, which is also called a symmetric or cyclic trapezium. Then, we examine the rhombus, which is a parallelogram the sides of which are of equal length and has reflectional symmetry along both diagonals.*

Keywords: Poly-Universe triangle, isosceles trapezium, rhombuses, quadrilateral

1 POLY-UNIVERSE TRIANGLE: DRAWING, CODING, CALCULATION OF PERIMETER

Let's introduce the following code to identify the elements: an element is indicated by the sequence of the letters R, B, G and Y (the initials of the colours), with the colours being placed in decreasing order of the areas of their fields. So RBGY is the element in which red is the base colour, blue is the colour of the largest field in the corner, the medium-sized one is green, and the smallest one is yellow.

Draw any element of the triangle set into your notebook (or on the task sheet) using a ruler and a pair of compasses. What is the name of this element in our code system? Calculate the perimeter of each triangle and that of the field in the middle.

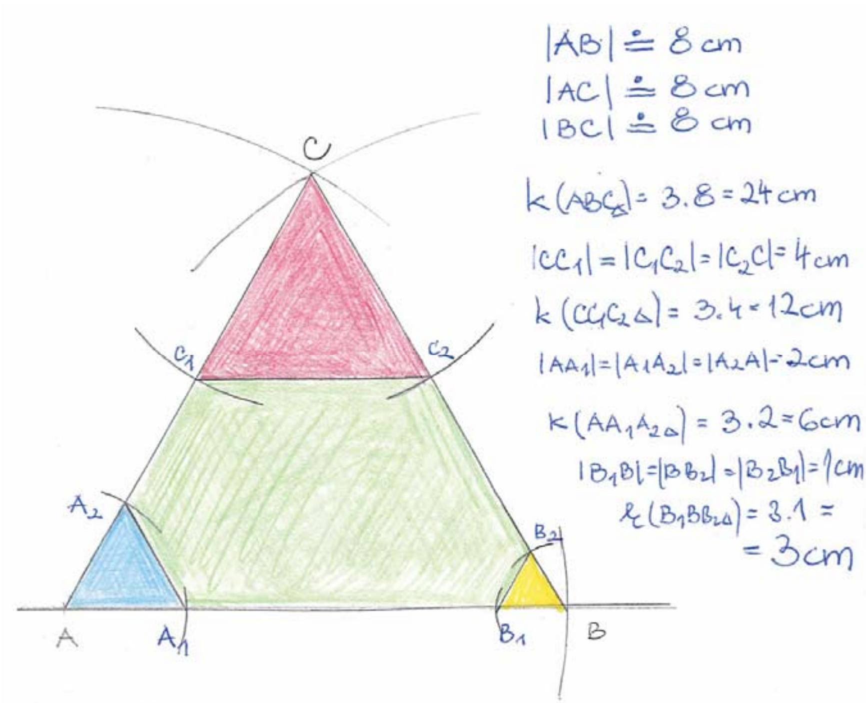

Figure 1: Freehand drawing of the Poly-Universe triangle

1.1 Code: GRBY

1.2 Perimeters: basic element 3 × 8 = 24; red field 3 × 4 = 12; blue field 3 × 2 = 6; yellow field 3 × 1 = 3.

The perimeter of the middle field is:

1.2.1 Routine solution: we measure the sides and add them up: 1 + 5 + 2 + 2 + 4 + 3 = 17.

1.2.2 A more sophisticated solution: we can fold out the common sides of the hexagon and the small triangles onto the sides of the basic triangle. This way we gain a complete side (8), 1 cm will be missing from another one (7) and the side of the hexagon remains (8 - 2 - 4 = 2). Thus, 8 + 7 + 2 = 17.

Recommendation: In this task, the students practice working with a ruler and a pair of compasses. They practise using the coding system of the shapes and practice calculating the perimeter. The calculation of the perimeter of the field in the middle is a differentiating task. It is useful to discuss different solutions so that students can gain the acknowledgement of their peers.

2 POLY-UNIVERSE TRIANGLE: COMPLETE COLOUR COMBINATION

The inventor developed the sets containing the following elements by using the formal and colour elements of the Polyuniverse[1] so that all of the elements are different. How many elements are there in the set? Why? How can it be calculated?

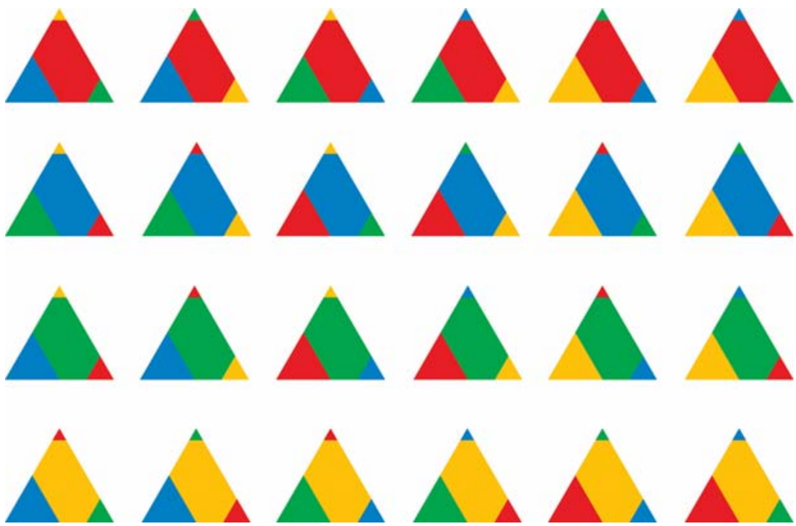

Figure 2: Colour combination arrangement of the Poly-Universe triangle

[1] Erasmus+ PUSE Study (Poly-Universe in School Education) 2017−1−HU01−KA201−035938 Project.

Solution: The number of the elements in each set presents itself through the simple permutation without repetition of the four colours in all of the three cases, i.e. each set contains 4 • 3 • 2 • 1= 4! = 24 elements.

3 HOW CAN QUADRILATERALS BE CONSTRUCTED?

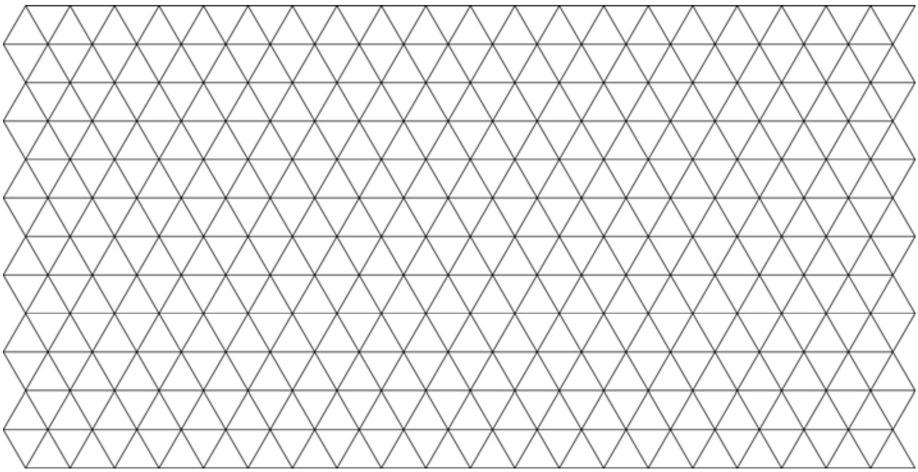

Figure 3: Students can draw the isosceles trapeziums and rhombii of different size on the triangle grid to preserve the information they have acquired and to apply this knowledge to solve other tasks.

3.1 Isosceles trapeziums

Description of the task: Construct isosceles trapeziums of different size. How many triangles did you use from one set of triangles to create the smallest and the largest possible isosceles trapeziums? Draw them on the triangle grid.

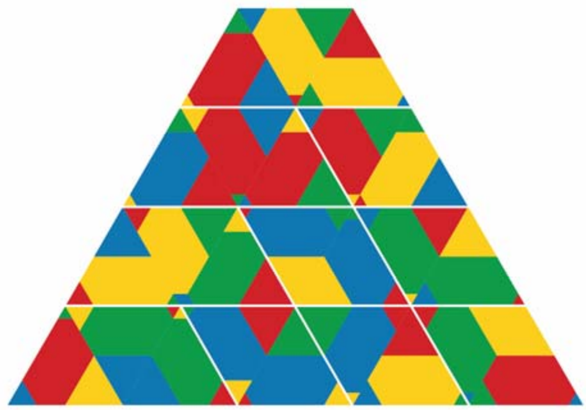

Figure 4: Drawn isosceles trapeziums of different size

Solution: We can see the largest possible isosceles trapezium in this photo, which was created using 24 elements of the triangle set. For the smallest possible trapezium, 3 elements are needed. We can see all the possible solutions along the white line segments: Trapeziums from 1 row of triangles: 3, 5, 7, 9; Trapeziums from 2 rows of triangles: 8, 12, 16; Trapeziums from 3 rows of triangles: 15, 21; Trapeziums from 4 rows of triangles: 24.

3.2 Rhombii

Descriptions of the task: Construct rhombii of different size. How many triangles did you use to create the smallest and the largest possible rhombii using one set of triangles? Draw them on the triangle grid.

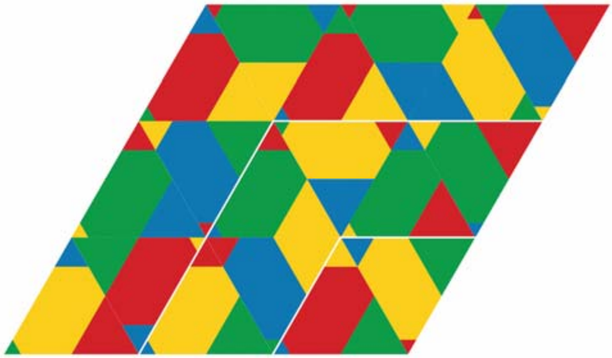

Figure 4: Drawed rhombuses of different size

Solution: 2 elements are needed for the smallest rhombus, then 8 elements for the medium-sized rhombus, and finally 18 elements are needed for the largest rhombus.

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AREA CALCULATION OF THE POLY-UNIVERSE TRIANGLE

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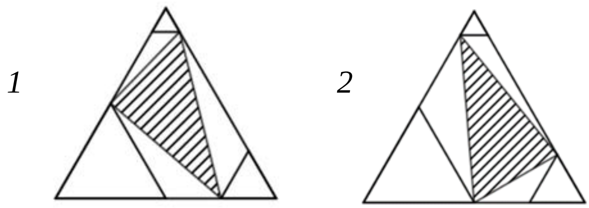
***Abstract**: The calculation of area and volume is almost the same age as Mathematics itself. In the following examples based on the Poly-Universe triangles, we will show the equality of the areas of different shapes with diverse and funny techniques and we intend to further reflect on our previous result (See PUSE Geometry[1] 140-143C Tasks).*

Keywords: Geometry, Symmetry, calculation of area

1

The question is: What proportion of the whole is the shaded part in the first and the second figure?



1.1 For the sake of simplicity, let the sides of the triangle be 8 units long. The area of an equilateral triangle with side length *a* is

$$A = \frac{a^2\sqrt{3}}{4}.$$

[1] Erasmus+ PUSE Study (Poly-Universe in School Education) 2017−1−HU01−KA201−035938 Project

$$A_{ABC} = 16\sqrt{3}$$

The area of the triangle EFD is determined by subtracting the area of the triangles ADF, DBE, and ECF from the area of the base triangle ABC. Since the heights of the triangles are equal to the heights of the corresponding equilateral triangles with sides of length 4, of 2, and of 1 unit,

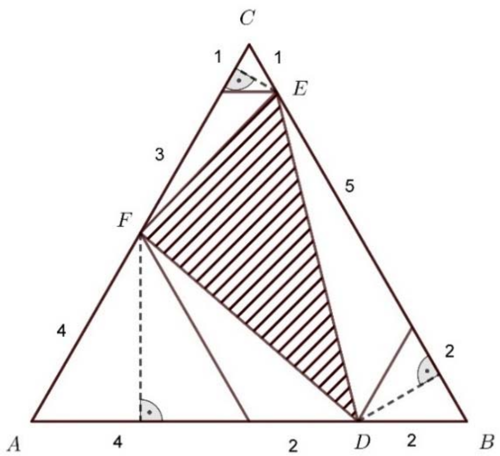
$$A_{ADF} = \frac{6 \cdot 2\sqrt{3}}{2} = 6\sqrt{3}$$

$$A_{BED} = \frac{7 \cdot \sqrt{3}}{2} = 3.5\sqrt{3}$$

$$A_{CEF} = \frac{4 \cdot \frac{\sqrt{3}}{2}}{2} = \sqrt{3}.$$

So $A_{DEF} = 16\sqrt{3} - 6\sqrt{3} - 3.5\sqrt{3} - \sqrt{3} = 5.5\sqrt{3}$

Therefore, the ratio of the shaded area to the whole is: $\frac{A_{DEF}}{A_{ABC}} = \frac{5.5\sqrt{3}}{16\sqrt{3}} = \frac{11}{32}$



1.2 The way of solution is the same in case of the second triangle as well: the area of the shaded triangle can be calculated by subtracting the area of the white triangles from the area of the base triangle.

$$A_{ABC} = 16\sqrt{3} \qquad A_{ADF} = \frac{7 \cdot 2\sqrt{3}}{2} = 7\sqrt{3}$$

$$A_{BED} = \frac{4 \cdot \sqrt{3}}{2} = 2\sqrt{3} \quad A_{CEF} = \frac{6 \cdot \frac{\sqrt{3}}{2}}{2} = 1.5\sqrt{3}.$$

So $A_{DEF} = 16\sqrt{3} - 7\sqrt{3} - 2\sqrt{3} - 1.5\sqrt{3} = 5.5\sqrt{3}$

Therefore, the ratio is $\frac{11}{32}$ again.

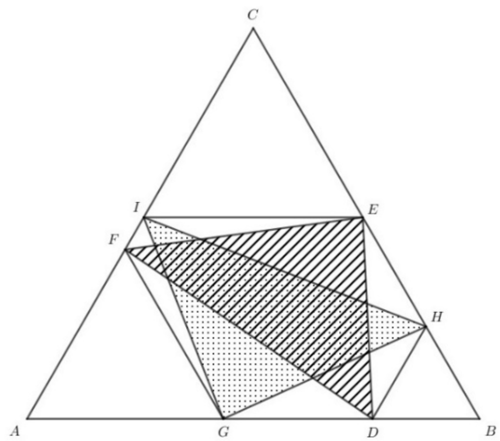
In the previous exercises, we have seen that when the lengths of the sides of triangles AGF, BHD, and CIE are one half, one quarter, and one eighth of the side of triangle ABC, the areas of triangles EDF and GHI
are the same (both $\frac{11}{32}$).

Is this statement true if the ratios of the sides of the triangles are not in progression 1: 2: 4: 8?

2

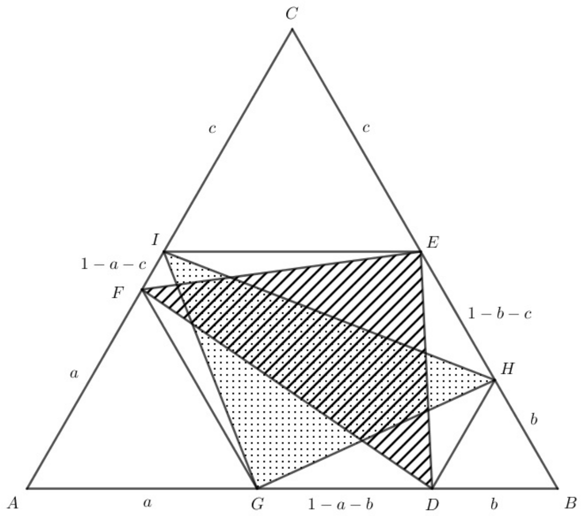
So, the new task is: An equilateral triangle ABC is given in which FG, DH, EI are secant segments parallel to the opposite sides respectively. Is it true that $T_{GHI} = T_{DEF}$?

For the sake of simplicity, let the length of the sides of triangle ABC be 1 unit; the side of triangle AFG is a; the side of triangle BHD is b; and the side of triangle CIE is c.



The areas of triangles DEF and GHI are determined by subtracting the areas of the corresponding triangles from the area of triangle ABC. We can calculate the area of the given triangles in many ways, now we will use the trigonometric area rule

$$A_{\Delta} = \frac{a \cdot b \cdot sin\gamma}{2} \dots$$



2.1 So

$$T_{DEF} = T_{ABC} - T_{ADF} - T_{BED} - T_{CFE} =$$

$$= \frac{1 \cdot 1 \cdot \sin 60°}{2} - \frac{a \cdot (1-b) \cdot \sin 60°}{2} - \frac{b \cdot (1-c) \cdot \sin 60°}{2} - \frac{c \cdot (1-a) \cdot \sin 60°}{2} =$$

$$= \frac{\sin 60°}{2} \cdot [1 - a(1-b) - b(1-c) - c(1-a)] =$$

$$= \frac{\sin 60°}{2} \cdot (1 - a - b - c + ab + bc + ac)$$

2.2 and

$$T_{GHI} = T_{ABC} - T_{AGI} - T_{BHG} - T_{CIH} =$$

$$= \frac{1 \cdot 1 \cdot \sin 60°}{2} - \frac{a \cdot (1-c) \cdot \sin 60°}{2} - \frac{b \cdot (1-a) \cdot \sin 60°}{2} - \frac{c \cdot (1-b) \cdot \sin 60°}{2} =$$

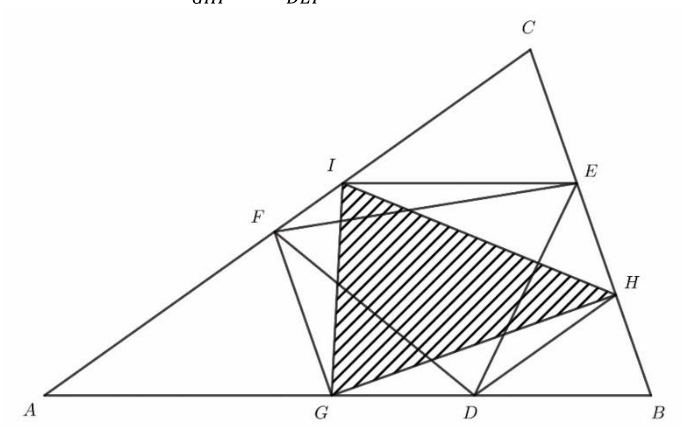
$$= \frac{\sin 60°}{2} \cdot [1 - a(1-c) - b(1-a) - c(1-b)] =$$

$$= \frac{\sin 60°}{2} \cdot [1 - a(1-c) - b(1-a) - c(1-b)] =$$
$$= \frac{\sin 60°}{2} \cdot (1 - a - b - c + ab + bc + ac)$$

So, the areas of the two triangles are the same.

3

3.1 A triangle ABC is given in which FG, DH, EI are secant segments parallel with the opposite sides. Is it true that $T_{GHI} = T_{DEF}$?



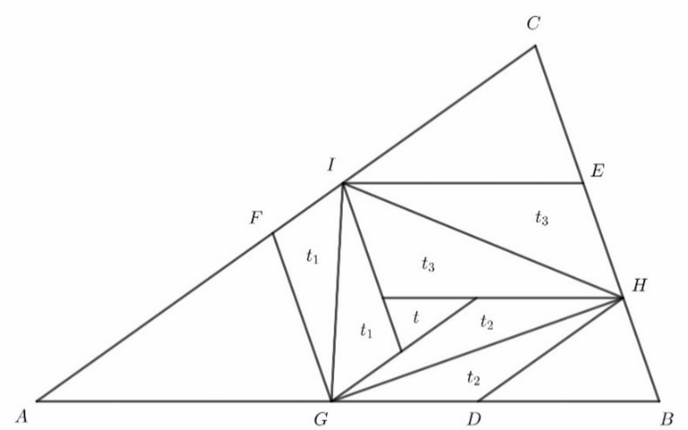
Let's consider the triangles AFG, BDH, and CIE as being the images of triangle ABC under dilation, the centres of which are the vertices A, B, and C, and the scale factors of which are a, b, and c. We know that the ratio of the areas is quadratic with the scale factor, so the ratio of the area of the triangle AFG to the area of triangle ABC $= a^2 : 1$. If the ratio of line segment AF to AC is a, then the ratio of line segment BD to AB is b, and the ratio of side AD to AB is *1-b*. Let's try to determine the ratio of the area of triangle AFD to the area of triangle ABC. We already know that AGF: ABC = a^2, and we also know that the heights of triangles AGF and ADF are the same and that the ratio of their bases is $a : 1\text{-}b$. Therefore, the ratio of their areas also changes with the ratio of their sides. That is, AGF: ADF = a^2: (1-b). So ADF : ABC $= a\,(1-b)$. The ratio of the areas of triangles BDE, CEF to the area of triangle ABC can be calculated similarly, and they are *b (1-c)* and *c (1-a)*, respectively.

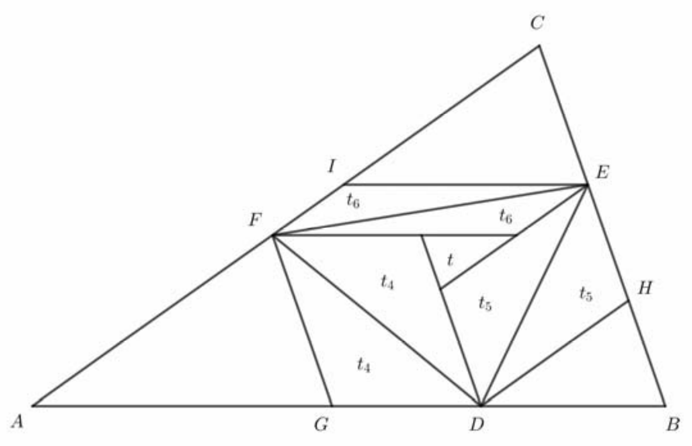
The areas of triangles DEF and GHI are determined by subtracting the areas of the corresponding triangles from the area of triangle ABC.

So $A_{DEF} = A_{ABC} - A_{ADF} - A_{BED} - A_{CFE} = 1 - a(1-b) - b(1-c) - c(1-a) = (1 - a - b - c + ab + ac + bc)$.

$A_{GHI} = A_{ABC} - A_{AGI} - A_{BHG} - A_{CIH} = 1 - a(1-c) - b(1-a) - c(1-b) = (1 - a - b - c + ab + ac + bc)$.

So, the areas of the two triangles are the same.





3.2 In a slightly different way: Let's consider hexagon FGDHEI. Draw lines parallel with the appropriate sides through points G, I, H, and then through points D, E, and F. This way the hexagon is divided into 7 triangles, the areas of which can be paired, because the diagonals of the parallelograms halve the areas of the quadrilaterals. The triangles in the middle of the two different figures are congruent because their sides are of equal length since their sides are $\mathrm{IE} - \mathrm{GD}, \mathrm{DH} - \mathrm{IF}, \mathrm{FG} - \mathrm{EH}$. Because the two hexagons are the same, so are their areas: $2t_1 + 2t_2 + 2t_3 + t = 2t_4 + 2t_5 + 2t_6 + t$; if we subtract *t* from both sides and divide by 2 then

$t_1 + t_2 + t_3 = t_4 + t_5 + t_6$; then adding t to both sides

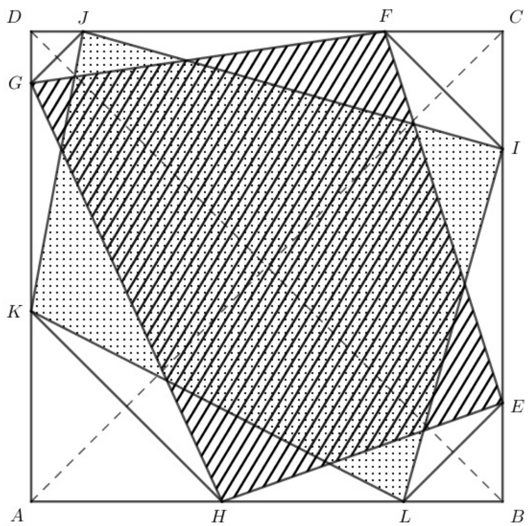
$t_1 + t_2 + t_3 + t = t_4 + t_5 + \mathrm{t}_6 + t$,

$A_{GHI} = A_{DEF}$;

So, the statement has been proved.

4

4.1 Is the statement true even if the task is based on a square instead of a triangle?



So, the new task is: A square ABCD is given in which LE, IF, JG and KH are secant segments parallel with the diagonals of the square.
Is it true that $T_{EFGH} = T_{IJKL}$?

Let the lengths of the following line segments be denoted in this way:

$\mathrm{AH} = \mathrm{AK} = a, \mathrm{BE} = \mathrm{BL} = b, \mathrm{CF} = \mathrm{CI} = c$, and $\mathrm{DG} = \mathrm{DJ} = d$.

We can assume that the length of the sides of the square is one unit. Calculate the area of quadrilateral EFGH in the following way:

$A_{EFGH} = A_{ABCD} - A_{AHG} - A_{BEH} - A_{CFE} - A_{DGF}$;

So $A_{EFGH} = 1 - \frac{a(1-d)}{2} - \frac{b(1-a)}{2} - \frac{c(1-b)}{2} - \frac{d(1-c)}{2}$;

Now calculate the area of quadrilateral IJKL in a similar way.

We get the following expression for this:

$A_{IJKL} = 1 - \frac{a(1-b)}{2} - \frac{b(1-c)}{2} - \frac{c(1-d)}{2} - \frac{d(1-a)}{2}$;

4.2 If we expand the brackets, we can see that the two expressions are the same, so the areas of the two quadrilaterals are the same as well.

Finally, one more idea. The same problem in the 3D space. A transformation of central similitude is applied to a tetrahedron ABCD with each vertex as a centre. The four diminished tetrahedra obtained are $AA_bA_cA_d$, $B_aBB_cC_d$, $C_aC_bCC_d$ and $D_aD_bD_cD$.

Given that these small tetrahedra are pairwise disjoint, prove that the volumes of tetrahedra $A_bB_cC_dD_a$, $A_bB_dD_cC_a$, $A_cC_bB_dD_a$, $A_cC_dD_bB_a$, $A_dD_bB_cC_a$ and $A_dD_cC_bB_a$ are equal.

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SYMMETRY: SCIENCE AND ART

BISYMMETRIC MATRICES AND SYMMETROLOGIC RESEARCHES IN GENETIC BIOMECHANICS

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Abstract: *Symmetrologic approaches play one of the foundation roles in mathematical natural sciences. The article is devoted to applications of bisymmetric (2^n*2^n)-matrices to model and study symmetrical features of DNA alphabets and also rules of percentages in content of long DNA sequences. Such bisymmetric matrices are known in mathematics as bisymmetric doubly stochastic matrices and as matrix representations of hypercomplex 2^n-dimensional numbers. The discovery of the importance of this type of matrices for modelling genetic structures provides new opportunities for symmetrologic applications and for the mutual enrichment of different sciences.*

Keywords: DNA, bisymmetric doubly stochastic matrix, unitary matrix, double number

1 INTRODUCTION

One of the creators of quantum mechanics P. Jordan correctly pointed out the following main difference between living cells and inanimate objects. Inanimate objects are governed by the average random motion of millions of particles, such that the motion of a single molecule has no influence whatsoever on the whole object. By contrast, the few molecules control dynamics of living cells and have dictatorial influence through quantum-level events that govern their motion and are amplified to influence the entire organism (McFadden, Al-Khalili, 2018).

Living bodies are ensembles of a huge number of molecules interrelated by quantum-mechanical, resonant, stochastic and other principles. These ensembles have an amazing ability to inherit their biological traits to their descendants. G.Mendel in his experiments on the crossing of organisms discovered that the inheritance of these characters occurs according to algebraic rules. These algebraic rules of polyhybrid crossbreeding are presented since 1906 in the form of Punnett squares. To explain the discovered rules, Mendel proposed the remarkable idea of binary-opposition forms of existence of inheritance factors: dominant and recessive forms, combinations of which are shown in Punnett squares. By these reasons Mendel can be considered as a pioneer in the field of algebraic biology, which is under intensive development in our days to study, first of all, the dictatorial influence of genetic molecules on the whole organism (Petoukhov, 2008; Petoukhov, He, 2010). All inherited physiological systems, as parts of a whole organism, must be structurally coupled with a genetic code for transmission to descendants in encoded form. Therefore, inherited macrostructures can bear the imprint of structural features of the genetic code. Taking this into account, genetic biomechanics is developing as a new scientific direction, where symmetrologic approaches play an important role (Petoukhov, 2001, 2003-2006, 2008, 2012, 2015, 2016a,b, 2018a,b, 2019a,b; Petoukhov, He, 2010). The presented article is devoted to some symmetrologic thematic researches related with DNA binary-opposition structures.

2 THE GENETIC CODE AND GENETIC MATRICES

In DNA molecules genetic information is written in sequences of 4 kinds of nucleobases: adenine A, cytosine C, guanine G, and thymine T. As known, the set of these 4 nucleobases A, C, G, and T is endowed with binary-opposition indicators:

- 1) in the double helix of DNA there are two complementary pairs of letters: the letters C and G are connected by three hydrogen bonds, and the letters A and T by two hydrogen bonds. Given the opposition indicators, one can represent C=G=1, A=T=0;

- 2) the two letters are keto molecules (G and T), and the other two are amino molecules (A and C). Given these opposition indicators, one can represent A=C=1, G=T=0.

Taking this into account, it is convenient to present DNA alphabets of 4 letters, 16 doublets and 64 triplets in the form of square tables, the columns of which are numbered in accordance with oppositional indicators “3 or 2 hydrogen bonds” (C=G=1, A=T=0), and the rows in accordance with oppositional indicators “amino or keto” (C=A=1, G=T=0). In such tables, all letters, doublets and triplets automatically occupy their strictly individual places. Fig. 1 shows the (2*2)-matrix of 4 nucleobases and the (8*8)-matrix of 64 triplets constructed by this way.

	1	0
1	G	T
0	C	A

	111	110	101	100	011	010	001	000
111	GGG	GGT	GTG	GTT	TGG	TGT	TTG	TTT
110	GGC	GGA	GTC	GTA	TGC	TGA	TTC	TTA
101	GCG	GCT	GAG	GAT	TCG	TCT	TAG	TAT
100	GCC	GCA	GAC	GAA	TCC	TCA	TAC	TAA
011	CGG	CGT	CTG	CTT	AGG	AGT	ATG	ATT
010	CGC	CGA	CTC	CTA	AGC	AGA	ATC	ATA
001	CCG	CCT	CAG	CAT	ACG	ACT	AAG	AAT
000	CCC	CCA	CAC	CAA	ACC	ACA	AAC	AAA

Figure 1: The square tables of DNA-alphabets of 4 nucleobases and 64 triplets with a strict arrangement of all components on the basis of binary-oppositional indicators of nucleobases A, C, G, and T.

These two tables (Fig. 1) are not only simple tables but they are members of the tensor family of matrices [G, T; C, A]$^{(n)}$: the third tensor (Kronecker) power of the matrix [C, A; G, T] gives the same arrangements of 64 triplets shown in Fig. 1. The genetic code is called a "degenerate code" because 64 triplets encode 20 amino acids and stop-codons so that several triplets can encode each amino acid at once and each triplet necessarily encodes only a single amino acid or a stop-codon. The (8*8)-matrix of 64 triplets (Fig. 1) was built formally without any mention of amino acids and stop-codons. How can these 20 amino acids and stop-codons be located in this matrix of 64 triplets? There are a huge number of possible options for the location and repetition of separate amino acids and stop-codons in 64 cells of this matrix. More precisely, the number of these options is much more than 10^{100} (for comparison, the entire time of the Universe existence is estimated in modern physics at 10^{17} seconds). Will this arrangement of amino acids be chaotic or will it suddenly turn out to be algebraically logical?

It occurs that Nature chosen by unknown reasons an algebraically logical option, whose analysis is important for revealing the structural organization of the informational foundations of living matter. Fig. 2 shows the real repetition and location of amino acids and stop-codons in the Vertebrate Mitochondrial Code, which is the most symmetrical among known dialects on the genetic code. This genetic code is called the most ancient and "ideal" in genetics (Frank Kamenetskii, 1997) (other dialects of the genetic code have small differences from this one).

	111	110	101	100	011	010	001	000
111	PRO	PRO	HIS	GLN	THR	THR	ASN	LYS
110	PRO	PRO	GLN	HIS	THR	THR	LYS	ASN
101	ARG	ARG	LEU	LEU	SER	STOP	ILE	MET
100	ARG	ARG	LEU	LEU	STOP	SER	MET	ILE
011	ALA	ALA	ASP	GLU	SER	SER	TYR	STOP
010	ALA	ALA	GLU	ASP	SER	SER	STOP	TYR
001	GLY	GLY	VAL	VAL	CYS	TRP	PHE	LEU
000	GLY	GLY	VAL	VAL	TRP	CYS	LEU	PHE

Figure 2: The location and repetition of 20 amino acids and 4 stop-codons in the matrix of 64 triplets $[C, A; G, T]^{(3)}$ (Fig. 1) for the Vertebrate Mitochondrial Code.

The location and repetition of amino acids and stop-codons in the matrix of 64 triplets have the symmetrologic feature (Fig. 2): each of sixteen (2*2)-sub-quadrants of this genetic matrix is bisymmetrical since each of its diagonals contain an identical kind of amino acids or stop-codons in its two cells. If each amino acid and stop-codon is represented by some characteristic parameter (for example, the number of carbon atoms in these organic formations or numbers of protons in its molecular structure, etc.), then a numerical (8*8)-matrix arises with bisymmetric (2*2)-sub-quadrants representing so-called double numbers $a+bj_1$ where $j_1^2 = +1$ (as known, double numbers are represented by bisymmetric (2*2)-matrices (Petoukhov, 2008; Petoukhov, He, 2010)).

The connection of the genetic code with double numbers, demonstrated by the matrix of 64 triplets in Fig. 2, supplements the following statement, presented in several previous publications (Petoukhov, 2008, 2016; Petoukhov, He, 2010, etc.). The genetic code is not just a mapping of one set of elements to other sets of elements by type, for example, of a phone book in which phone numbers encode names of people. But the genetic code is inherently an algebraic code, akin to those algebraic codes that are used in modern communication theory for noise-immune transmission of information. The authors believe that algebraic features of the genetic code participate in the noise-immune properties of this code and of the whole genetic system.

About the theme of matrix genetics and algebraic biology, this article draws attention also to another important case of bisymmetric matrices known in algebra and its applications. We are talking about so-called doubly stochastic matrices, which, by definition, are square matrices whose entries are non-negative and their sum in each row and each column is unity. Doubly stochastic matrices have long been known and important for many problems of game theory and linear programming: optimizing access to resources, forming winning coalitions, logistics, growth models in oncology, etc. (Bapat, Raghavan, 1997; Prasolov, 1994). This can allow you to transfer some already known mathematical results from game theory and other fields into the field of models of algebraic biology. It opens new abilities to study mathematically a possible role of optimization principles and biological principles of natural selection in biological genetic evolution. As far as the authors know, the first mention about a connection of doubly stochastic matrices with a special kind of genetic matrices was made by Prof. Matthew He in the book (Petoukhov, He, 2010). This article shows new data on relation of the genetic system with doubly stochastic matrices.

The bisymmetric doubly stochastic matrices are associated with unitary matrices that are important for quantum informatics, quantum mechanics, noise-immune coding of information, etc.; they are also used in matrix genetics (Petoukhov, 2008, 2018a; Petoukhov, He, 2010). The following known theorem states this connection (Prasolov, 1994): if a square ($n*n$)-matrix $M=\|m_{ij}\|$ is unitary then an ($n*n$)-matrix $B=\|b_{ij}\|$, where $b_{ij} = |m_{ij}|^2$, is doubly stochastic. Bisymmetric doubly stochastic matrices can be interpreted as a special class of hypercomplex double numbers having non-negative coordinates, the sum of which is unity. One example of connections of bisymmetric doubly stochastic matrices with the features of the genetic systems is the following. Using doubly stochastic matrices allows modeling some phenomenological rules of sequences of hydrogen bonds in long DNA revealed and described early in (Petoukhov, 2018b, 2019a). More precisely, in any DNA double helix, the complementary letters C-G and A-T are connected by 3 and 2 hydrogen bonds, respectively; it is representing as C = G = 3 and A = T = 2. Therefore, long DNA contains millions of chains of hydrogen bonds from the numbers 3 and 2 of type 33223223233... It turns out that such hydrogen chains of long DNA follow a single phenomenological rule: knowing in a long DNA chain 33223223233... the percentages q and p of numbers 3 and 2 correspondingly ($q + p = 1$), you can predict in it with good accuracy the percentage of all doublets 33, 32, 23 and 22 (as, %32, %23, %22), triplets 333, 332, 323, 322, 233, 232, 223, 222, tetraplets and pentaplets (that is n-plets with n = 2, 3, 4, 5) based on the tensor family of bisymmetric doubly stochastic matrices $[q, p; p, q]^{(n)}$ representing 2^n-dimensional double numbers with nonnegative coordinates whose sum is unity (here n = 1, 2, 3, 4, 5; j_1, j_2, j_3, … are imaginary units of these hypercomplex numbers with their property $j_k^2=+1$): the matrix $[q, p; p, q]$ represents 2-dimensional double numbers $q+pj_1$; $[q, p; p, q]^{(2)}$ represents 4-dimensional double numbers $qq+qpj_1+pqj_2+ppj_3$; $[q, p; p, q]^{(3)}$ represents 8-dimensional double numbers $qqq+qqpj_1+qpqj_2+qppj_3+pqqj_4+pqpj_5+ppqj_6+pppj_7$, etc. The coordinates of these double numbers efficiently model the percentages of n-plets of hydrogen bonds 3 and 2. For example, the coordinate *qp* models the percentage of doublets 32; the coordinate *pqq* models the percentage of triplets 233, etc. Fig. 3 shows one of genomic examples confirming this rule. More examples are shown in (Petoukhov, 2018a,b, 2019a,b).

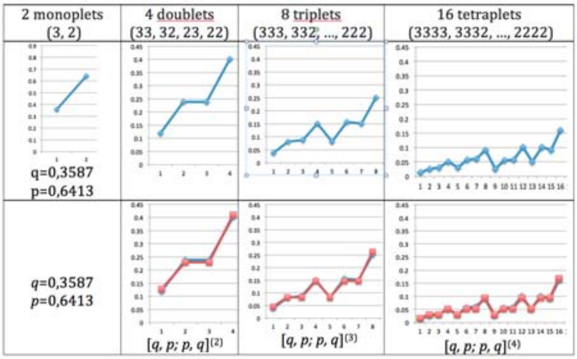

Figure 3: Upper graphical row: phenomenological percentages of *n*-plets of hydrogen bonds 3 and 2 in DNA sequences of the first chromosome of a plant Arabidopsis thaliana. Each of points in graphs shows percentage value of an appropriate member of alphabets of *n*-plets. Bottom graphical row: model percentage values of these percentages as coordinates of 2^n-dimensional double numbers represented by bisymmetric doubly stochastic matrices $[q, p; p, q]^{(n)}$. Initial data about this DNA sequence, having 30427671 bp, were taken from the GenBank: https://www.ncbi.nlm.nih.gov/genome/4.

Tensor families of bisymmetrical doubly stochastic matrices are useful for modeling percentage features not only of long DNA sequences but also, for example, of long literary Russian texts in the case of phonetic binary grouping letters of Russian alphabets and representing such texts as appropriate binary sequences (Petoukhov, 2019a,b). In living nature, information at different levels of organization is transmitted in the form of chain-like sequences or texts: DNA nucleotide chains, protein chains of amino acids, pulse sequences in neurons, literary and musical texts ... One can think that the whole informatics in living bodies is built on symmetrologic templates or archetypes of genetic informatics. Here researches have access to the rich theme of the archetypes of C. Jung, V. Pauli, the ancient Chinese book "I-Ching" and Eastern medicine. Authors believe that using bisymmetric matrices will be useful to study this wide topic.

3 SOME CONCLUDING REMARKS

Symmetrologic approaches play one of the foundation roles in mathematical natural sciences (Darvas, 2007, 2018; Darvas, Petoukhov, 2020). Bisymmetric matrices are useful mathematical tool to include biology in the field of these sciences. Studying symmetries in the organization of the genetic coding system, we come to the algebraic patterns in it and new knowledge about the dictatorial effect of DNA on the body. The authors urge symmetrologists to pay attention to the symmetrologic organization of the genetic coding system for advancing in the knowledge of informational foundations of living bodies and the principles of combining parts into the whole organism.

Regarding data of our article, it seems important that bisymmetric doubly stochastic matrices are related with unitary matrices as it was mentioned above. In particular, all calculations in quantum computers are conducted based on unitary matrices (Nielsen, Chuang, 2010), which represent transformations of turns and mirror reflections in vector spaces. Taking into account published data about relations of genetic systems with quantum informatics and unitary matrices (Petoukhov, Petukhova, 2020; Petoukhov, Petukhova, Svirin, 2019), one can speculate that unitary matrices are also a basic tool to model physiological manage processes in living bodies. In particular, it seems to be no coincidence that the kinematic articular schemes of human and animal bodies are built specifically on unitary transformations of turns and mirror reflections.

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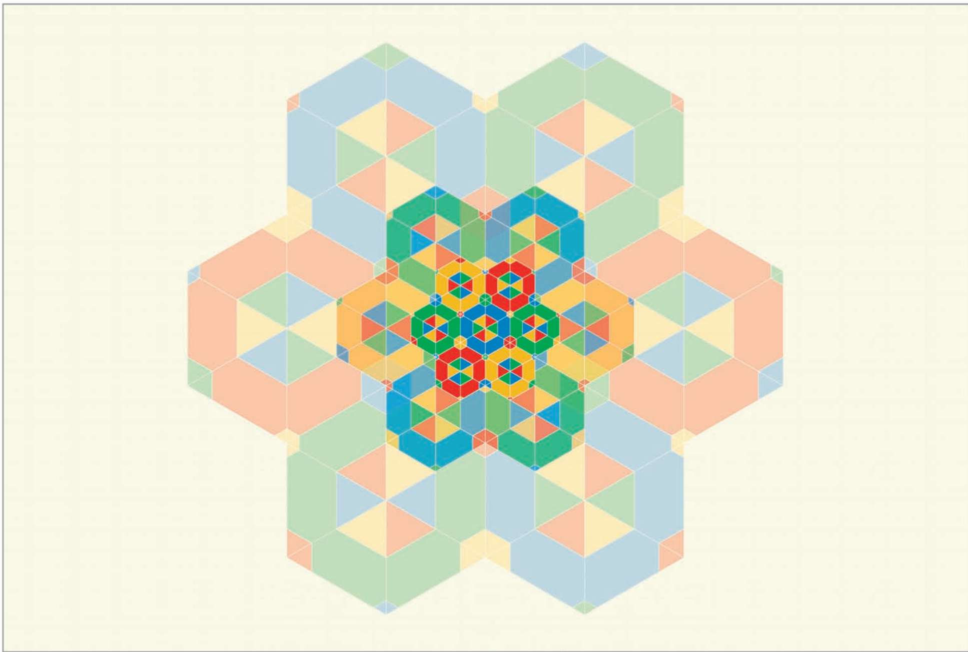

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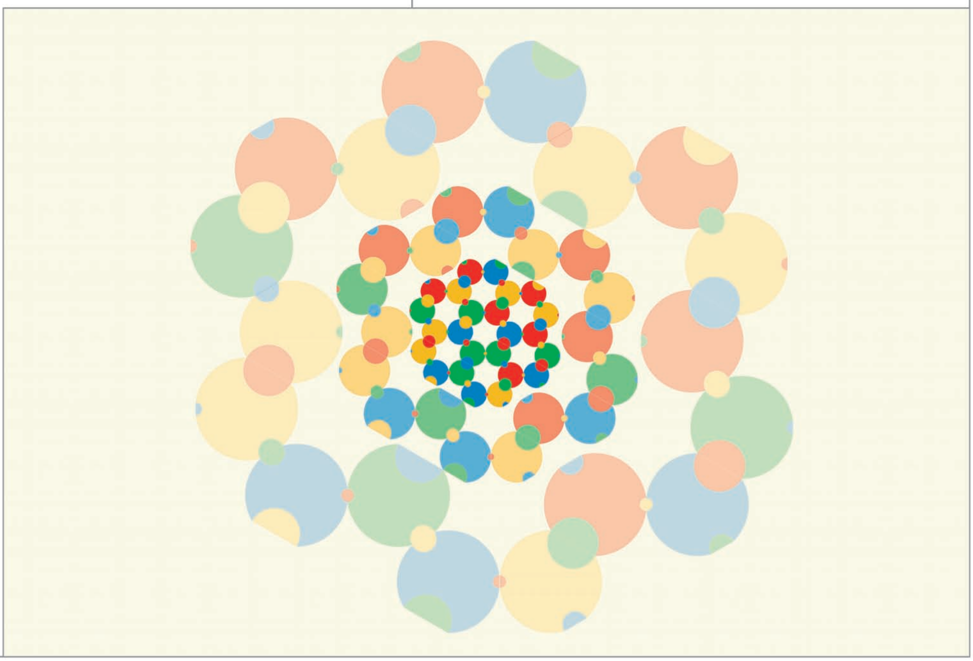
Symmetry



SYMMETRION
Budapest,Hungary
http://symmetry.hu